Name:

ID:

- This is a timed examination. You are allowed 120 minutes to finish it.
- There are 12 questions worth 10 points each.
- This exam covers Chapters 4 and 5 in the textbook.

- No calculators or other devices may be used.
- No books or notes other than a 3×5 inch index card of notes and formulas are permitted, nor any collaboration.
- Read the statement of each problem carefully.
- Be sure to ask questions if anything is unclear.
- Show all your work for full credit.
- Your ability to make your solution clear will be part of your grade.

1. Find the general solutions of the following second-order ODE:

$$y'' + 2y' + 5y = 0\,.$$

$$r^2 + 2r + 5 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4\cdot 5}}{2}$$

$$= \frac{-2 \pm \sqrt{-16}}{2}$$

$$= \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

General solution is $y = Ae^{-x}\cos 2x + Be^{-x}\sin 2x$

2. Find the solution of the following initial value problem:

$$y'' - 6y' + 5y = 0; \qquad y(0) = 1,\ y'(0) = 1\,.$$

$r^2 - 6r + 5 = 0$

$(r-1)(r-5) = 0$

$r = 1, 5$

General solution is $y = Ae^{x} + Be^{5x}$

$1 = y(0) = Ae^{0} + Be^{5\cdot 0} = A + B$

$1 = y'(0) = Ae^{0} + 5Be^{5\cdot 0} = A + 5B$

$4B = 0 \Rightarrow B = 0,\ A = 1$

So sol. of i.v.p. is $y = 1\cdot e^{x} + 0e^{5x} = e^{x}$.

3. Find the general solution of the following ODE:

$$y'' - 4y' + 4y = 2x\,.$$

$r^2 - 4r + 4 = 0$

$(r-2)(r-2) = 0$

$r = 2, 2.$

Sol. of homogeneous is $y = Ae^{2x} + Bxe^{2x}$

For a particular solution, guess $y_0 = \alpha x + \beta$

$y_0' = \alpha$

$y_0'' = 0$

$0 - 4(\alpha) + 4(\alpha x + \beta) = 2x$

$4\alpha x + (4\beta - 4\alpha) = 2x$

$\left.\begin{matrix} 4\alpha = 2 \\ 4\beta - 4\alpha = 0 \end{matrix}\right\} \begin{matrix} \alpha = \frac{1}{2} \\ \beta = \frac{1}{2} \end{matrix}$

$y_0 = \frac{1}{2}x + \frac{1}{2}$

Sol. of ODE is

$y = \left(\frac{1}{2}x + \frac{1}{2}\right) + Ae^{2x} + Bxe^{2x}.$

4. Find the solution of this initial value problem:

$$y'' + y = x^2; \qquad y(0) = 1,\ y'(0) = 1\,.$$

Solution to homogeneous is

$$y = A\cos x + B\sin x$$

Guess $y_0 = \alpha x^2 + \beta x + \gamma$ for a particular solution.

$$(\alpha x^2 + \beta x + \gamma)'' + (\alpha x^2 + \beta x + \gamma) = x^2$$

$$2\alpha \qquad + \alpha x^2 + \beta x + \gamma = x^2$$

$$\alpha x^2 + \beta x + (2\alpha + \gamma) = x^2$$

$$\alpha = 1$$

$$\beta = 0$$

$$2\alpha + \gamma = 0 \Rightarrow \gamma = -2.$$

$$y_0 = x^2 - 2.$$

$$y = (x^2 - 2) + A\cos x + B\sin x$$

$$1 = y(0) = -2 + A \Rightarrow A = 3$$

$$1 = y'(0) = B$$

Sol. of i.v.p. is $y = (x^2 - 2) + 3\cos x + 1\sin x$

5. The equation $xy'' + y' = 0$ has the trivial solution $y_1(x) = 1$. Find a second linearly independent solution y_2 and then find the general solution.

Seek a second solution $y_2 = vy_1$. Then

$$v = \int \frac{1}{1^2} e^{-\int \frac{1}{x} dx} dx = \int \frac{1}{x} dx = \ln|x|$$

So $y_2 = \ln|x| \cdot 1 = \ln|x|$.

The general solution is

$$y = A \cdot 1 + B \cdot \ln|x| = A + B \ln|x|$$

6. The ODE

$$y^{(6)} + 5y^{(4)} - 2y^{(3)} + 6y''8y' - 8y = 0$$

has associated polynomial

$$r^6 - 5r^4 - 2r^3 + 6r^2 + 8r - 8 = (r-1)^2(r^2+2r+2)(r^2-4) = 0\,.$$

Find the general solution.

$(r-1)^2 = 0 \Rightarrow r = 1, 1$

$r^2 + 2r + 2 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$

$r^2 - 4 = 0 \Rightarrow r = \pm 2$

General solution is

$y = Ae^x + Bxe^x + Ce^{-x}\cos x + De^{-x}\sin x$
$\quad + Ee^{2x} + Fe^{-2x}$

7. Find the general solution of $y''' + 4y'' + 3y' = 1 + \cos x$.

$$y''' + 4y'' + 3y' = 0$$
$$r^3 + 4r^2 + 3r = 0$$
$$r(r^2 + 4r + 3) = 0$$
$$r(r+3)(r+1) = 0$$
$$r = 0,\ -3,\ -1.$$
$$y = Ae^{0x} + Be^{-3x} + Ce^{-x} = A + Be^{-3x} + Ce^{-x}$$

is solution of homogeneous.

For a particular solution guess

$$y_0 = \alpha\cos x + \beta\sin x + \gamma x$$
$$y_0' = -\alpha\sin x + \beta\cos x + \gamma$$
$$y_0'' = -\alpha\cos x - \beta\sin x$$
$$y_0''' = \alpha\sin x - \beta\cos x$$

$$(\alpha\sin x - \beta\cos x) + 4(-\alpha\cos x - \beta\sin x) + 3(-\alpha\sin x + \beta\cos x + \gamma) = 1 + \cos x$$

$$(\alpha - 4\beta - 3\alpha)\sin x + (-\beta - 4\alpha + 3\beta)\cos x + 3\gamma = 1 + \cos x$$

$$\left.\begin{aligned} \alpha - 4\beta - 3\alpha &= 0 \\ -\beta - 4\alpha + 3\beta &= 1 \\ 3\gamma &= 1 \end{aligned}\right\} \quad \beta = \tfrac{1}{10},\ \alpha = -\tfrac{1}{5},\ \gamma = \tfrac{1}{3}$$

So $y_0 = -\frac{1}{5}\cos x + \frac{1}{10}\sin x + \frac{1}{3}x$

General solution is

$$y = -\frac{1}{5}\cos x + \frac{1}{10}\sin x + \frac{1}{3}x + A + Be^{-3x} + Ce^{-x}$$

8. Use the method of undetermined coefficients to find the general solution of the differential equation

$$y'' - 3y' + 2y = x\,.$$

$$y'' - 3y' + 2y = 0$$

$$r^2 - 3r + 2 = 0$$

$$(r-1)(r-2) = 0$$

$$r = 1,\ 2.$$

Sol. of homogeneous is $y = Ae^{x} + Be^{2x}$

Guess $y_0 = \alpha x + \beta$

$$y_0' = \alpha$$

$$y_0'' = 0$$

$$0 - 3\alpha + 2(\alpha x + \beta) = x$$

$$2\alpha x + (2\beta - 3\alpha) = x$$

$$\left.\begin{array}{l} 2\alpha = 1 \\ 2\beta - 3\alpha = 0 \end{array}\right\} \begin{array}{l} \alpha = \frac{1}{2} \\ \beta = \frac{3}{4} \end{array}$$

$$y_0 = \frac{1}{2}x + \frac{3}{4}.$$

General solution is

$$y = \left(\frac{1}{2}x + \frac{3}{4}\right) + Ae^{x} + Be^{2x}.$$

9. (a) State Kepler's Laws of Planetary Motion.
   (b) State Newton's Law of Universal Gravitation.

(a) 1. The orbit of each planet is an ellipse with the sun at one focus.

2. The segment from the center of the sun to the center of an orbiting planet sweeps out area at a constant rate.

3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of the elliptical orbit, with the same constant of proportionality for any planet.

(b) If the sun has mass M and the planet has mass m then the gravitational force exerted by the sun on the planet is

$$-\frac{GmM}{r^2}\underline{\underline{u}}$$

where G is a universal gravitational constant, $\underline{\underline{u}}$ is a unit vector pointing from the sun to the planet, and r is the distance from center of sun to center of planet.

10. Find the power series for $\frac{1}{(1+x)^2}$.

We know that

$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \cdots$$

So

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - + \cdots$$

Taking the derivative,

$$-\frac{1}{(1+x)^2} = -1 + 2x - 3x^2 + 4x^3 - + \cdots$$

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + - \cdots$$

11. Find a power series solution of the form $\sum_j a_j x^j$ for the following ODE:

$$y' = 2xy\,.$$

Identify the function it represents.

Let $y = \sum_{j=0}^{\infty} a_j x^j$.

$$\sum_{j=1}^{\infty} j a_j x^{j-1} = 2x \sum_{j=0}^{\infty} a_j x^j$$

$$\sum_{j=-1}^{\infty} (j+2) a_{j+2} x^{j+1} - \sum_{j=0}^{\infty} 2 a_j x^{j+1} = 0$$

$$a_1 x^0 + \sum_{j=0}^{\infty} \left[(j+2) a_{j+2} - 2 a_j \right] x^{j+1} = 0$$

$$a_1 = 0$$

$$(j+2) a_{j+2} - 2 a_j = 0, \quad j \geq 0$$

$$a_{j+2} = \frac{2}{j+2} a_j, \quad j \geq 0$$

Let

$$a_0 = C$$

$$a_1 = 0$$

$$a_2 = C$$

$$a_3 = 0$$

$$a_4 = \frac{2 a_2}{4} = \frac{C}{2}$$

$$a_5 = 0$$

$$a_6 = \frac{2 a_4}{6} = \frac{C}{3 \cdot 2}$$

$$a_7 = 0$$

$$a_8 = \frac{2 a_6}{8} = \frac{C}{4 \cdot 3 \cdot 2}$$

$$y = C \sum_{j=0}^{\infty} \frac{1}{j!} x^{2j} = C e^{x^2}.$$

12. Find the power series solution to the initial value problem

$$y'' - xy' - y = 1, \qquad y(0) = 1, y'(0) = 1\,.$$

$$y = \sum_{j=0}^{\infty} a_j x^j$$

$$\sum_{j=2}^{\infty} j(j-1) a_j x^{j-2} - x \sum_{j=1}^{\infty} j a_j x^{j-1} - \sum_{j=0}^{\infty} a_j x^j = 1$$

$$\sum_{j=0}^{\infty} (j+2)(j+1) a_{j+2} x^j - \sum_{j=1}^{\infty} j a_j x^j - \sum_{j=0}^{\infty} a_j x^j = 1$$

$$2a_2 x^0 - a_0 x^0 + \sum_{j=1}^{\infty} \left[(j+2)(j+1) a_{j+2} - j a_j - a_j \right] x^j = 1$$

$$2a_2 - a_0 = 1, \quad (j+2)(j+1) a_{j+2} - (j+1) a_j = 0, \quad j \geq 1$$

$$a_{j+2} = \frac{a_j}{j+2}.$$

Let $a_0 = C$, $a_1 = D$

$$a_2 = \frac{a_0 + 1}{2} = \frac{C+1}{2}$$

$$a_3 = \frac{a_1}{3} = \frac{D}{3}$$

$$a_4 = \frac{a_2}{4} = \frac{C+1}{4 \cdot 2}$$

$$a_5 = \frac{a_3}{5} = \frac{D}{5 \cdot 3}$$

$$a_6 = \frac{a_4}{6} = \frac{C+1}{6 \cdot 4 \cdot 2}$$

The initial conditions tell us That $a_0 = C = 1$, $a_1 = D = 1$

So

$$y = 1 + 1x + 1x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{15}x^5 + \frac{1}{24}x^6 + \cdots$$