SOLUTIONS TO PRACTICE MT3

1. $x^2y'' + x(x+1)y' - y = 0$

$$y'' + \frac{x+1}{x}y' - \frac{1}{x^2}y = 0$$

$x \cdot \frac{x+1}{x} = x+1$ regular

$x^2 \cdot \left(-\frac{1}{x^2}\right) = -1$ regular.

So 0 is a regular singular point

$$f(m) = m(m-1) + 1m - 1 = m^2 - m + m - 1 = 0$$

$$m^2 - 1 = 0$$

$$m = \pm 1$$

are the valid values for m.

2. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} -x\,dx = \frac{1}{\pi}\left(\frac{-x^2}{2}\right)\Big]_{-\pi}^{\pi} = 0$

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} -x\cos jx\,dx = \frac{1}{\pi}\frac{\sin jx}{j}(-x)\Big]_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin jx}{j}\,dx$$

$$= (0-0) - \frac{1}{\pi}\frac{\cos jx}{j^2}\Big]_{-\pi}^{\pi} = 0$$

$$b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} -x\sin jx\,dx = \frac{1}{\pi}x\frac{\cos jx}{j}\Big]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos jx}{j}\,dx$$

$$= \frac{1}{\pi}\pi\frac{(-1)^j}{j} + \frac{1}{\pi}\pi\frac{(-1)^j}{j} - \frac{1}{\pi}\frac{\sin jx}{j^2}\Big]_{-\pi}^{\pi}$$

$$= \frac{2(-1)^j}{j} + (0-0) = \frac{2(-1)^j}{j}.$$

So the Fourier series is

$$\sum_{j=1}^{\infty} \frac{2(-1)^j}{j} \sin jx,$$

3. Let $\tilde{g}(x) = \begin{cases} 2x, & 0 \le x \le \pi \\ -2x, & -\pi \le x < 0, \end{cases}$

Then g is even. So its Fourier series has only cosine terms.

Thus

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{g}(x)\,dx = \frac{2}{\pi}\int_0^{\pi} 2x\,dx = \frac{2}{\pi} x^2 \Big]_0^{\pi}$$
$$= 2\pi.$$

For $j \ge 1$,

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{g}(x) \cos jx\,dx$$
$$= \frac{2}{\pi}\int_0^{\pi} 2x \cos jx\,dx = \frac{4}{\pi}\, \frac{\sin jx}{j} x \Big]_0^{\pi} - \frac{4}{\pi}\int_0^{\pi} \frac{\sin jx}{j}\,dx$$
$$= (0-0) + \frac{4}{\pi}\, \frac{\cos jx}{j^2}\Big]_0^{\pi}$$
$$= \frac{4}{\pi j^2}\left[(-1)^j - 1\right].$$

So Fourier series is

$$\frac{2\pi}{2} + \sum_{j=1}^{\infty} \frac{4}{\pi j^2}\left[(-1)^j - 1\right] \cos jx,$$

4. $a_0 = \frac{1}{L}\int_{-L}^{L} -x\,dx = \frac{1}{3}\int_{-3}^{3} -x\,dx = \frac{1}{3}\left(\frac{-x^2}{2}\right)\Big]_{-3}^{3} = 0$

$j \geq 1$, $a_j = \frac{1}{3}\int_{-3}^{3}(-x)\cos\frac{j\pi x}{3}\,dx = \frac{1}{3}\left[(-x)\frac{\sin j\pi x/3}{j\pi/3}\right]_{-3}^{3}$

$$+ \frac{1}{3}\cdot\frac{3}{j\pi}\int_{-3}^{3}\sin j\pi x/3\,dx$$

$$= (0-0) - \frac{1}{j\pi}\frac{\cos j\pi x/3}{j\pi}\Big]_{-3}^{3}$$

$$= -\frac{1}{j^2\pi^2}(-1)^j + \frac{1}{j^2\pi^2}(-1)^j = 0$$

$b_j = \frac{1}{3}\int_{-3}^{3}(-x)\sin\frac{j\pi x}{3}\,dx = \frac{1}{3}\left[(-x)\frac{-\cos j\pi x/3}{j\pi/3}\right]_{-3}^{3}$

$$- \frac{1}{3}\cdot\frac{3}{j\pi}\int_{-3}^{3}\cos j\pi x/3\,dx$$

$$= \frac{1}{j\pi}\left[(-3)\left(-(-1)^j\right) + 3(-1)^j\right]$$

$$+ \frac{9}{j^2\pi^2}\sin j\pi x/3\Big]_{-3}^{3}$$

$$= \frac{6}{j\pi}(-1)^j + (0-0)$$

So Fourier series is

$$\sum_{j=1}^{\infty}\frac{6}{j\pi}(-1)^j \sin j\pi x/3.$$

5. $$\frac{1}{p^4-4p^2} = \frac{1}{p^2(p^2-4)} = \frac{1}{p^2(p-2)(p+2)}$$

$$= \frac{A}{p^2} + \frac{B}{p-2} + \frac{C}{p+2}$$

$$= \frac{A(p-2)(p+2) + Bp^2(p+2) + Cp^2(p-2)}{p^2(p-2)(p+2)}$$

$$= \frac{A(p^2-4) + B(p^3+2p^2) + C(p^3-2p^2)}{p^2(p-2)(p+2)}$$

$$= \frac{p^3(B+C) + p^2(A+2B-2C) + p\cdot 0 - 4A}{p^2(p-2)(p+2)}$$

$\therefore$ $B+C=0$

$A+2B-2C=0$

$-4A=1 \Rightarrow A=-\frac{1}{4}$

$B+C=0 \Rightarrow 2B+2C=0$

$2B-2C=\frac{1}{4} \Rightarrow 2B-2C=\frac{1}{4}$

$4B=\frac{1}{4}$

$B=\frac{1}{16} \Rightarrow C=-\frac{1}{16}$

So

$$\frac{1}{p^4-4p^2} = \frac{-1/4}{p^2} + \frac{1/16}{p-2} - \frac{1/16}{p+2}$$

$$\Rightarrow L^{-1}\left(\frac{1}{p^4-4p^2}\right) = -\frac{1}{4}x + \frac{1}{16}e^{2x} - \frac{1}{16}e^{-2x}.$$

6. $y'' - 2y' + y = 0$, $y(0) = 0,\ y'(0) = 3$

$L[y''] - 2L[y'] + L[y] = 0$

$[p^2 L[y] - p \cdot 0 - 3] - 2[p\,L[y] - 0] + L[y] = 0$

$(p^2 - 2p + 1)\,L[y] = 3$

$$L[y] = \frac{3}{p^2 - 2p + 1} = \frac{3}{(p-1)^2}.$$

So $y = 3e^x \cdot x$.

7. $f(t) = t,\ g(t) = e^{2t}$

$$f * g(x) = \int_0^x (x-t)e^{2t}\,dt$$

$$= x\int_0^x e^{2t}\,dt - \int_0^x t e^{2t}\,dt$$

$$= x\,\frac{e^{2t}}{2}\Big]_0^x - \left[t\frac{e^{2t}}{2}\Big]_0^x - \int_0^x \frac{e^{2t}}{2}\,dt\right]$$

$$= \frac{xe^{2x}}{2} - \frac{x}{2} - \frac{xe^{2x}}{2} + 0 + \frac{e^{2t}}{4}\Big]_0^x$$

$$= -\frac{x}{2} + \frac{e^{2x}}{4} - \frac{1}{4}.$$

8. $y'' + y' - 6y = t,\ y(0) = 0,\ y'(0) = 0.$

$z(p) = p^2 + p - 6$

$$L[A] = \frac{1}{p(p^2+p-6)} = \frac{1}{p(p+3)(p-2)} = \frac{A}{p} + \frac{B}{p+3} + \frac{C}{p-2}$$

$$= \frac{A(p+3)(p-2) + B(p)(p-2) + Cp(p+3)}{p(p+3)(p-2)}$$

$$= \frac{A(p^2+p-6) + B(p^2-2p) + C(p^2+3p)}{p(p+3)(p-2)}$$

$$= \frac{p^2(A+B+C) + p(A-2B+3C) - 6A}{p(p+3)(p-2)}$$

So
$$A + B + C = 0$$
$$A - 2B + 3C = 0$$
$$-6A = 1 \Rightarrow A = -\frac{1}{6}$$

$$-\frac{1}{6} + B + C = 0 \Rightarrow -\frac{1}{2} + 3B + 3C = 0$$
$$-\frac{1}{6} - 2B + 3C = 0 \qquad -\frac{1}{6} - 2B + 3C = 0$$

$$-\frac{1}{3} + 5B = 0$$
$$B = \frac{1}{15}$$
$$C = \frac{1}{10}$$

Thus
$$L[A] = \frac{-1/6}{p} + \frac{1/15}{p+3} + \frac{1/10}{p-2} \Rightarrow A = -\frac{1}{6} + \frac{1}{15}e^{-3x} + \frac{1}{10}e^{2x}$$

$$y(t) = \int_0^t \left(-\frac{1}{6} + \frac{1}{15}e^{-3(t-\tau)} + \frac{1}{10}e^{2(t-\tau)}\right) \cdot 1\, d\tau$$

$$+ 0 \cdot A(t)$$

$$= -\frac{1}{6}\tau + \frac{1}{15}e^{-3t}\frac{e^{3\tau}}{3} + \frac{1}{10}e^{2t}\frac{e^{-2\tau}}{-2}\Bigg]_0^t$$

$$= \left(-\frac{1}{6}t + \frac{1}{45} - \frac{1}{20}\right) - \left(0 + \frac{1}{45}e^{-3t} - \frac{1}{20}e^{2t}\right)$$

$$= -\frac{1}{6}t - \frac{1}{36} - \frac{1}{45}e^{-3t} + \frac{1}{20}e^{2t}.$$

9. Let $f(x) = (\cos x) \cdot (\sin x)$. Then

$$f(-x) = (\cos -x) \cdot (\sin -x) = (\cos x) \cdot (-\sin x)$$
$$= -(\cos x) \cdot (\sin x) = -f(x).$$

So f is odd.

10. $$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin jx \cos kx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(j+k)x + \sin(j-k)x \, dx$$

$$= \frac{1}{2} \cdot \frac{-1}{j+k}\cos(j+k)x + \frac{1}{2} \cdot \frac{-1}{j-k}\cos(j-k)x \Bigg]_{-\pi}^{\pi}$$

$$= 0.$$

11. $F(p) = \int_0^\infty e^{-px} f(x)\,dx$

$$\frac{d}{dp} F(p) = \frac{d}{dp} \int_0^\infty e^{-px} f(x)\,dx$$

$$= \int_0^\infty \frac{d}{dp} e^{-px} f(x)\,dx$$

$$= \int_0^\infty e^{-px} \cdot (-x \cdot f(x))\,dx$$

$$= L[-x f(x)].$$

12. $L[xe^x] = -L[-xe^x]$

$$= -\frac{d}{dp} L[e^x]$$

$$= -\frac{d}{dp}\left(\frac{1}{p-1}\right)$$

$$= -\left(\frac{-1}{(p-1)^2}\right) = \frac{1}{(p-1)^2}.$$