Math 217 Spring, 2019
Krantz May 3, 2019

FINAL EXAM

General Instructions: Read the statement of each problem carefully. You do not need to show any work– simply mark your answers on the provided answer (scantron) card.

Be sure to ask questions if anything is unclear. This exam is worth 100 points. Each problem is weighted equally (100/13 points).

1. Solve the first-order linear equation

$$y' + 2x \cdot y = 2\pi x\,.$$

(a) $y = 2\pi + Ce^{-x^2}$
(b) $y = 1 + Ce^{x^2}$
(c) $y = -\pi/2 + Ce^{-x^2}$
(d) $y = 1/2 + Ce^{-x^2/2}$
(e) $y = \pi + Ce^{-x^2}$

Solution:
Multiply the equation by the integrating factor $e^{\int 2x dx}$:

$$e^{x^2}\left(y' + 2xy\right) = 2\pi x e^{x^2}$$

$$\left(e^{x^2} y\right)' = 2\pi x e^{x^2}$$

$$\int \left(e^{x^2} y\right)' dx = \int 2\pi x e^{x^2} dx$$

$$e^{x^2} y = \pi e^{x^2} + C$$

$$y = \pi + Ce^{-x^2}$$

2. Let x and y be positive. The differential equation

$$-\frac{\ln xy}{M(x,y)}\,\frac{dy}{dx} = 1$$

can be solved using the method of exact equations if and only if, for some differentiable function $h(x)$,

(a) $M(x,y) = \dfrac{y}{x} + h(x)$

(b) $M(x,y) = \dfrac{1}{x} + h(x)$

(c) $M(x,y) = \ln xy + h(x)$

(d) $M(x,y) = \dfrac{y}{x}$

(e) $M(x,y) = \dfrac{x}{y} + h(x)$

<u>Solution:</u>
For an equation of the form

$$M(x,y)dx + N(x,y)dy = 0$$

we can use the method of exact equations if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Namely, rewriting the equation as

$$M(x,y)dx + \ln xy dy = 0$$

we must have

$$\frac{\partial M}{\partial y} = \frac{1}{xy}\,x = \frac{1}{x}$$

or equivalently

$$M(x,y) = \frac{y}{x} + h(x).$$

3. The differential equation

$$y'' - 4y' + 3y = 0$$

has general solution given by

(a) $y = Ae^{-x} + Be^{-2x}$

(b) $y = Ae^{-4x} + Be^{3x}$

(c) $y = Ae^{x} + Be^{3x}$

(d) $y = Ae^{4x} + Be^{-3x}$

(e) $y = Ae^{2x} + Be^{-2x}$

<u>Solution:</u>
The auxiliary equation is $r^2 - 4r + 3$ which factors as $(r-1)(r-3)$ and has roots $r = 1, 3$. Thus, the general solution is

$$y = Ae^{x} + Be^{3x}.$$

4. Use the method of undetermined coefficients to find the general solution of the differential equation

$$y'' - 4y' + 3y = 3x^2 - 5x - 2\,.$$

(a) $y = (x^2) + Ae^{-x} + Be^{-2x}$

(b) $y = (x^2 + x) + Ae^{x} + Be^{3x}$

(c) $y = (x) + Ae^{-4x} + Be^{3x}$

(d) $y = (4x^2) + Ae^{4x} + Be^{-3x}$

(e) $y = (x^2 - x - 2) + Ae^{2x} + Be^{-2x}$

<u>Solution:</u>
Guess a particular solution of the form

$$y_0(x) = ax^2 + bx + c.$$

Computing derivatives and substituting into the original equation we obtain

$$2a - 4(2ax + b) + 3(ax^2 + bx + c) = 3x^2 - 5x - 2.$$

Equating like coefficients we see

$$3a = 3$$

$$-8a + 3b = -5$$

$$2a - 4b + 3c = -2$$

which gives $a = 1$, $b = 1$, and $c = 0$. Hence,

$$y_0(x) = x^2 + x.$$

Adding the general solution from the previous problem, our final solution is

$$y = \left(x^2 + x\right) + Ae^{x} + Be^{3x}.$$

5. One solution of

$$x^2y'' - 2y = 0$$

is $y = x^2$. Find another.

(a) $\dfrac{1}{x^2}$

(b) $\dfrac{3}{x^2}$

(c) $-x^2$

(d) $-\dfrac{1}{x}$

(e) $-\dfrac{1}{3x}$

<u>Solution:</u>
First, put the equation in standard form:

$$y'' + 0y' - \frac{2}{x^2}y = 0.$$

We know if $y_1(x)$ is a solution, another solution $y_2(x)$ is given by

$$y_2(x) = \left(\int \frac{1}{y_1^2}\, e^{-\int p(x)dx}\, dx\right) y_1(x).$$

Substituting for $y_1(x)$ and noting that $p(x) = 0$, we have

$$\begin{aligned} y_2(x) &= \left(\int \frac{1}{x^4}\, dx\right) x^2 \\ &= -\frac{1}{3x}. \end{aligned}$$

6. The power series solution to

$$y' - xy = 0$$

is

(a) Ce^{-x^2}

(b) Ce^{x}

(c) Ce^{-x}

(d) Ce^{Cx}

(e) $Ce^{x^2/2}$

<u>Solution:</u> Substituting $y = \sum_{n=0}^{\infty} a_n x^n$ into the equation we get

$$\sum_{n=1}^{\infty}(n)a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$a_1 + \sum_{n=1}^{\infty}\left((n+1)a_{n+1} - a_{n-1}\right)x^n = 0$$

Writing out a few terms we see that for all $j \geq 0$,

$$a_{2j+1} = 0$$

$$a_{2j} = \frac{C}{2^j(j!)}$$

where $a_0 = C$ is free. Our final solution is

$$\sum_{j=0}^{\infty} \frac{C}{2^j(j!)} x^{2j} = C \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{x^2}{2}\right)^j = Ce^{x^2/2}.$$

7. For $\lambda > 0$, the eigenvalues λ_n and the eigenfunctions y_n for the equation $y'' + \lambda y = 0$ for the boundary conditions $y(0) = 0$, $y(\sqrt{\lambda}) = 0$ are

(a) $\lambda_n = n^2\pi^2, \;\; y_n = \sin\sqrt{\lambda} n\pi x$

(b) $\lambda_n = n\pi, \;\; y_n = \sin\sqrt{n\pi} x$

(c) $\lambda_n = n^2\pi, \;\; y_n = \sin\lambda\pi x$

(d) $\lambda_n = n\pi, \;\; y_n = \sin n^2\pi^2 x$

(e) $\lambda_n = n^2\pi^2, \;\; y_n = \sin\lambda n\pi x$

<u>Solution:</u>
The solution to the differential equation is given by

$$y = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda x}.$$

Given, $y(0) = 0$, we deduce $A = 0$. Given $y(\sqrt{\lambda}) = 0$, we deduce $B\sin\lambda = 0$, i.e. $\lambda = n\pi$. Thus,

$$\lambda_n = n\pi$$

and

$$y_n = \sin\sqrt{n\pi}\; x.$$

8. What is the Fourier series of

$$f(x) = \begin{cases} -1 & \text{if} \quad -\pi \le x < 0 \\ 0 & \text{if} \quad 0 \le x \le \pi \end{cases} \quad ?$$

(a) $1 + \sum_{j=1}^{\infty} \frac{1}{\pi(j+1)} \cdot (-1 + (-1)^j) \sin jx$

(b) $-\frac{1}{2} + \sum_{j=1}^{\infty} \frac{1}{\pi j} \left(1 - (-1)^j\right) \sin jx$

(c) $\frac{1}{3} + \sum_{j=1}^{\infty} \frac{1}{2\pi j} \cdot (-1 + (-1)^j) \sin jx$

(d) $\frac{1}{2} + \sum_{j=1}^{\infty} \frac{2}{\pi j} \cdot (-1 + (-1)^j) \sin jx$

(e) $-\frac{1}{2} + \sum_{j=1}^{\infty} \frac{1}{3\pi j} \cdot (-1 + (-1)^j) \sin jx$

<u>Solution:</u>

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{0} (-1) dx \\ &= -1 \\ a_j &= \frac{1}{\pi} \int_{-\pi}^{0} (-1) \cos jx \; dx \\ &= -\frac{1}{\pi} \sin jx / j \Big|_{-\pi}^{0} \\ &= 0 \\ b_j &= \frac{1}{\pi} \int_{-\pi}^{0} (-1) \sin jx \; dx \\ &= \frac{1}{j\pi} \cos jx \Big|_{-\pi}^{0} \\ &= \frac{1}{\pi j} \left(1 - (-1)^j\right) \end{aligned}$$

Thus, the Fourier series is $-\frac{1}{2} + \sum_{j=1}^{\infty} \frac{1}{\pi j} \left(1 - (-1)^j\right) \sin jx$.

9. The sine series expansion of $\cos(x)$ on $[0, \pi]$ is

(a) $\frac{4}{\pi} \sum_{j=1}^{\infty} \frac{j}{8j-1} \sin 2jx$

(b) $\frac{8}{\pi} \sum_{j=1}^{\infty} \frac{j}{4j^2-1} \sin 2jx$

(c) $\frac{4}{\pi} \sum_{j=1}^{\infty} \frac{4j^2-1}{j} \sin 2jx$

(d) $\frac{1}{\pi} \sum_{j=1}^{\infty} \frac{j}{j-1} \sin 2jx$

(e) $\frac{8}{\pi} \sum_{j=1}^{\infty} \frac{4j^2-1}{j^2} \sin 2jx$

<u>Solution:</u>
Recalling that $\cos x \sin nx = \frac{1}{2}(\sin(n-1)x + \sin(n+1)x)$, we find that

$$b_n = \frac{2}{\pi} \int_0^{\pi} \left(\frac{1}{2} (\sin((n-1)x) + \sin((n+1)x)) \right) dx$$

which gives $b_1 = 0$ and

$$b_n = \frac{2n}{\pi} \left(\frac{1 + (-1)^n}{n^2 - 1} \right)$$

for $n > 1$. Accordingly, we have

$$b_{2j-1} = 0 b_{2j} \qquad = \frac{8j}{\pi(4j^2-1)}.$$

Thus, on $[0, \pi]$, we have

$$\cos x = \frac{8}{\pi} \sum_{j=1}^{\infty} \frac{j}{4j^2-1} \sin 2jx.$$

10. The solution of the Dirichlet problem on the unit disc with boundary data function

$$f(\theta) = \begin{cases} 0 & \text{if} \quad 0 \le \theta < \pi \\ 1 & \text{if} \quad \pi \le \theta \le 2\pi \end{cases}$$

is

(a) $w(r,\theta) = \dfrac{1}{2} + \displaystyle\sum_{j=1}^{\infty} \frac{1}{j\pi}\left((-1)^j - 1\right) r^j \sin j\theta$

(b) $w(r,\theta) = \dfrac{1}{2} + \displaystyle\sum_{j=1}^{\infty} \frac{4}{j\pi}\left((-1)^j + 1\right) r^{j+1} \sin j\theta$

(c) $w(r,\theta) = \dfrac{1}{2} + \displaystyle\sum_{j=1}^{\infty} \frac{-2}{j\pi}\left((-1)^{j+1} + 1\right) r^j \sin j\theta$

(d) $w(r,\theta) = \dfrac{1}{2} + \displaystyle\sum_{j=1}^{\infty} \frac{2}{j\pi}\left((-1)^j + 1\right) r^{j+1} \sin j\theta$

(e) $w(r,\theta) = \dfrac{1}{2} + \displaystyle\sum_{j=1}^{\infty} \frac{-4}{j\pi}\left((-1)^j + 1\right) r^j \sin j\theta$

<u>Solution:</u>

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{\pi}^{2\pi} (1)\cos 0 \; dx = \frac{1}{\pi}\; x|_{\pi}^{2\pi} = 1 \\
a_j &= \frac{1}{\pi}\int_{\pi}^{2\pi} (1)\cos jx \; dx = \frac{1}{\pi}\sin jx/j|_{\pi}^{2\pi} = 0 \;\; (j > 0) \\
b_j &= \frac{1}{\pi}\int_{\pi}^{2\pi} (1)\sin jx \; dx \\
&= -\frac{1}{\pi}\cos jx/j|_{\pi}^{2\pi} \\
&= -\frac{1}{\pi}\left(\frac{1^j}{j} - \frac{(-1)^j}{j}\right) = \frac{1}{j\pi}\left((-1)^j - 1\right)
\end{aligned}$$

Thus, the solution is

$$w(r,\theta) = \frac{1}{2} + \sum_{j=1}^{\infty} \frac{1}{j\pi}\left((-1)^j - 1\right) r^j \sin j\theta.$$

11. Using Laplace transforms, the solution to

$$y'' - y = 2019x$$

with initial conditions $y(0) = y'(0) = 0$ is

(a) $y = 2018(\sinh x - 1)$

(b) $y = 2019 \sinh x$

(c) $y = 2019(\sinh x - x)$

(d) $y = 2020 \sinh x$

(e) $y = 2020(\sinh x - 1)$

<u>Solution:</u>
Taking the Laplace transform of both sides gives

$$p^2 L[y] - py(0) - y'(0) - L[y] = \frac{2019}{p^2}$$

$$\begin{aligned} L[y] &= \frac{2019}{(p^2-1)p^2} \\ &= \frac{2019}{p^2-1} - \frac{2019}{p^2} \end{aligned}$$

Taking the inverse Laplace transform gives

$$y = 2019(\sinh x - x)$$

12. Using Laplace transforms, the solution to the integral equation

$$y(x) = \delta(x) + e^x - \int_0^x e^{x-t} y(t)\ dt$$

is

(a) $y(x) = \delta(x) - x$

(b) $y(x) = \delta(x) - 1$

(c) $y(x) = \delta(x)$

(d) $y(x) = \delta(x) - x^2$

(e) $y(x) = \delta(x) + x$

<u>Solution:</u>
Applying the Laplace transform to both sides we get

$$L[y] = L[\delta(x)] + L[e^x] - L[e^x]L[y].$$

Solving for $L[y]$ gives

$$\begin{aligned} L[y] &= \frac{L[\delta(x)] + L[e^x]}{1 + L[e^x]} \\ &= \frac{1 + L[e^x]}{1 + L[e^x]} \\ &= 1 \end{aligned}$$

This has inverse Laplace transform of

$$y(x) = \delta(x).$$

13. The impulsive response for the differential equation

$$y'' + y = f(t)$$

is

(a) $h(t) = \cos t$

(b) $h(t) = \sin t$

(c) $h(t) = \sin 2t$

(d) $h(t) = \cos 2t$

(e) $h(t) = \tan t$

<u>Solution:</u>
Recall that for the impulsive response, $f(t) = \delta(t)$ and $h(t)$ is given by the inverse Laplace transform of the reciprocal of the auxilliary equation. Thus,

$$\begin{aligned} h(t) &= L^{-1}\left[\frac{1}{p^2+1}\right] \\ &= \sin t. \end{aligned}$$