1. Solve the first-order linear equation

\[ y' + 2x \cdot y = 2\pi x. \]

(a) \( y = 2\pi + Ce^{-x^2} \)
(b) \( y = 1 + Ce^{x^2} \)
(c) \( y = -\pi/2 + Ce^{-x^2} \)
(d) \( y = 1/2 + Ce^{-x^2/2} \)
(e) \( y = \pi + Ce^{-x^2} \)

**Solution:**
Multiply the equation by the integrating factor \( e^{\int 2x \, dx} \).

\[ e^{x^2} (y' + 2xy) = 2\pi x e^{x^2} \]

\[ \left( e^{x^2} y \right)' = 2\pi x e^{x^2} \]

\[ \int \left( e^{x^2} y \right)' \, dx = \int 2\pi x e^{x^2} \, dx \]

\[ e^{x^2} y = \pi e^{x^2} + C \]

\[ y = \pi + Ce^{-x^2} \]
2. Let $x$ and $y$ be positive. The differential equation

$$-\frac{\ln xy}{M(x,y)} \frac{dy}{dx} = 1$$

can be solved using the method of exact equations if and only if, for some differentiable function $h(x)$,

(a) $M(x, y) = \frac{y}{x} + h(x)$
(b) $M(x, y) = \frac{1}{x} + h(x)$
(c) $M(x, y) = \ln xy + h(x)$
(d) $M(x, y) = \frac{y}{x}$
(e) $M(x, y) = \frac{x}{y} + h(x)$

Solution:

For an equation of the form

$$M(x, y) dx + N(x, y) dy = 0$$

we can use the method of exact equations if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$  

Namely, rewriting the equation as

$$M(x, y) dx + \ln xy dy = 0$$

we must have

$$\frac{\partial M}{\partial y} = \frac{1}{xy} x = \frac{1}{x}$$

or equivalently

$$M(x, y) = \frac{y}{x} + h(x).$$
3. The differential equation

\[ y'' - 4y' + 3y = 0 \]

has general solution given by

(a) \( y = Ae^{-x} + Be^{-2x} \)
(b) \( y = Ae^{-4x} + Be^{3x} \)
(c) \( y = Ae^{x} + Be^{3x} \)
(d) \( y = Ae^{4x} + Be^{-3x} \)
(e) \( y = Ae^{2x} + Be^{-2x} \)

Solution:
The auxiliary equation is \( r^2 - 4r + 3 \) which factors as \( (r - 1)(r - 3) \) and has roots \( r = 1, 3 \). Thus, the general solution is

\[ y = Ae^{x} + Be^{3x}. \]
4. Use the method of undetermined coefficients to find the general solution of the differential equation

\[ y'' - 4y' + 3y = 3x^2 - 5x - 2. \]

(a) \( y = (x^2) + Ae^{-x} + Be^{-2x} \)
(b) \( y = (x^2 + x) + Ae^x + Be^{3x} \)
(c) \( y = (x) + Ae^{-4x} + Be^{3x} \)
(d) \( y = (4x^2) + Ae^{4x} + Be^{-3x} \)
(e) \( y = (x^2 - x - 2) + Ae^{2x} + Be^{-2x} \)

Solution:

Guess a particular solution of the form

\[ y_0(x) = ax^2 + bx + c. \]

Computing derivatives and substituting into the original equation we obtain

\[ 2a - 4(2ax + b) + 3(ax^2 + bx + c) = 3x^2 - 5x - 2. \]

Equating like coefficients we see

\[ 3a = 3 \]
\[ -8a + 3b = -5 \]
\[ 2a - 4b + 3c = -2 \]

which gives \( a = 1, \ b = 1, \) and \( c = 0. \) Hence,

\[ y_0(x) = x^2 + x. \]

Adding the general solution from the previous problem, our final solution is

\[ y = (x^2 + x) + Ae^x + Be^{3x}. \]
5. One solution of
\[ x^2 y'' - 2y = 0 \]
is \( y = x^2 \). Find another.

(a) \( \frac{1}{x^2} \)  
(b) \( \frac{3}{x^2} \)  
(c) \( -x^2 \)  
(d) \( -\frac{1}{x} \)  
(e) \( -\frac{1}{3x} \)

Solution:
First, put the equation in standard form:
\[ y'' + 0y' - \frac{2}{x^2}y = 0. \]

We know if \( y_1(x) \) is a solution, another solution \( y_2(x) \) is given by
\[ y_2(x) = \left( \int \frac{1}{y_1^2} e^{-\int p(x) dx} \, dx \right) y_1(x). \]

Substituting for \( y_1(x) \) and noting that \( p(x) = 0 \), we have
\[ y_2(x) = \left( \int \frac{1}{x^2} \, dx \right) x^2 \]
\[ = -\frac{1}{3x}. \]
6. The power series solution to

\[ y' - xy = 0 \]

is

(a) \( Ce^{-x^2} \)
(b) \( Ce^{x} \)
(c) \( Ce^{-x} \)
(d) \( Ce^{Cx} \)
(e) \( Ce^{x^2/2} \)

**Solution:** Substituting \( y = \sum_{n=0}^{\infty} a_n x^n \) into the equation we get

\[
\sum_{n=1}^{\infty} (n) a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0
\]

\[
a_1 + \sum_{n=1}^{\infty} ((n+1) a_{n+1} - a_{n-1}) x^n = 0
\]

Writing out a few terms we see that for all \( j \geq 0, \)

\[ a_{2j+1} = 0 \]

\[ a_{2j} = \frac{C}{2^j (j!)} \]

where \( a_0 = C \) is free. Our final solution is

\[
\sum_{j=0}^{\infty} \frac{C}{2^j (j!)} x^{2j} = C \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{x^2}{2}\right)^j = Ce^{x^2/2}.
\]
7. For $\lambda > 0$, the eigenvalues $\lambda_n$ and the eigenfunctions $y_n$ for the equation $y'' + \lambda y = 0$ for the boundary conditions $y(0) = 0$, $y(\sqrt{\lambda}) = 0$ are

(a) $\lambda_n = n^2 \pi^2$, $y_n = \sin \sqrt{\lambda} n \pi x$
(b) $\lambda_n = n \pi$, $y_n = \sin \sqrt{n \pi} x$
(c) $\lambda_n = n^2 \pi$, $y_n = \sin \lambda n \pi x$
(d) $\lambda_n = n \pi$, $y_n = \sin n^2 \pi^2 x$
(e) $\lambda_n = n^2 \pi^2$, $y_n = \sin \lambda n \pi x$

Solution:
The solution to the differential equation is given by

$$y = A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x.$$  

Given, $y(0) = 0$, we deduce $A = 0$. Given $y(\sqrt{\lambda}) = 0$, we deduce $B \sin \lambda = 0$, i.e. $\lambda = n \pi$. Thus,

$$\lambda_n = n \pi$$

and

$$y_n = \sin \sqrt{n \pi} x.$$
8. What is the Fourier series of

\[ f(x) = \begin{cases} 
-1 & \text{if } -\pi \leq x < 0 \\
0 & \text{if } 0 \leq x \leq \pi 
\end{cases} \]

(a) \[ 1 + \sum_{j=1}^{\infty} \frac{1}{\pi(j+1)} \cdot (-1 + (-1)^j) \sin jx \]

(b) \[ -\frac{1}{2} + \sum_{j=1}^{\infty} \frac{1}{\pi j} (1 - (-1)^j) \sin jx \]

c) \[ \frac{1}{3} + \sum_{j=1}^{\infty} \frac{1}{2\pi j} \cdot (-1 + (-1)^j) \sin jx \]

d) \[ \frac{1}{2} + \sum_{j=1}^{\infty} \frac{2}{\pi j} \cdot (-1 + (-1)^j) \sin jx \]

e) \[ -\frac{1}{2} + \sum_{j=1}^{\infty} \frac{1}{3\pi j} \cdot (-1 + (-1)^j) \sin jx \]

Solution:

\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{0} (-1) \, dx \]
\[ = -1 \]

\[ a_j = \frac{1}{\pi} \int_{-\pi}^{0} (-1) \cos jx \, dx \]
\[ = -\frac{1}{\pi} \sin jx igg|_{-\pi}^{0} \]
\[ = 0 \]

\[ b_j = \frac{1}{\pi} \int_{-\pi}^{0} (-1) \sin jx \, dx \]
\[ = \frac{1}{j\pi} \cos jx igg|_{-\pi}^{0} \]
\[ = \frac{1}{\pi j} (1 - (-1)^j) \]

Thus, the Fourier series is \( -\frac{1}{2} + \sum_{j=1}^{\infty} \frac{1}{\pi j} (1 - (-1)^j) \sin jx \).
9. The sine series expansion of \( \cos(x) \) on \([0, \pi]\) is

(a) \[
\frac{4}{\pi} \sum_{j=1}^{\infty} \frac{j}{8j - 1} \sin 2jx
\]

(b) \[
\frac{8}{\pi} \sum_{j=1}^{\infty} \frac{j}{4j^2 - 1} \sin 2jx
\]

(c) \[
\frac{4}{\pi} \sum_{j=1}^{\infty} \frac{4j^2 - 1}{j} \sin 2jx
\]

(d) \[
\frac{1}{\pi} \sum_{j=1}^{\infty} \frac{j}{j - 1} \sin 2jx
\]

(e) \[
\frac{8}{\pi} \sum_{j=1}^{\infty} \frac{4j^2 - 1}{j^2} \sin 2jx
\]

**Solution:**
Recalling that \( \cos x \sin nx = \frac{1}{2} (\sin (n-1)x + \sin (n+1)x) \), we find that

\[
b_n = \frac{2}{\pi} \int_0^\pi \left( \frac{1}{2} (\sin ((n-1)x) + \sin ((n+1)x)) \right) dx
\]

which gives \( b_1 = 0 \) and

\[
b_n = \frac{2n}{\pi} \left( \frac{1 + (-1)^n}{n^2 - 1} \right)
\]

for \( n > 1 \). Accordingly, we have

\[
b_{2j-1} = 0b_{2j} = \frac{8j}{\pi(4j^2 - 1)}.
\]

Thus, on \([0, \pi]\), we have

\[
\cos x = \frac{8}{\pi} \sum_{j=1}^{\infty} \frac{j}{4j^2 - 1} \sin 2jx.
\]
10. The solution of the Dirichlet problem on the unit disc with boundary data function

\[ f(\theta) = \begin{cases} 
0 & \text{if } 0 \leq \theta < \pi \\
1 & \text{if } \pi \leq \theta \leq 2\pi 
\end{cases} \]

is

(a) \[ w(r, \theta) = \frac{1}{2} + \sum_{j=1}^{\infty} \frac{1}{j\pi} ((-1)^j - 1) r^j \sin j\theta \]

(b) \[ w(r, \theta) = \frac{1}{2} + \sum_{j=1}^{\infty} \frac{4}{j\pi} ((-1)^j + 1) r^{j+1} \sin j\theta \]

(c) \[ w(r, \theta) = \frac{1}{2} + \sum_{j=1}^{\infty} \frac{-2}{j\pi} ((-1)^{j+1} + 1) r^j \sin j\theta \]

(d) \[ w(r, \theta) = \frac{1}{2} + \sum_{j=1}^{\infty} \frac{2}{j\pi} ((-1)^j + 1) r^{j+1} \sin j\theta \]

(e) \[ w(r, \theta) = \frac{1}{2} + \sum_{j=1}^{\infty} \frac{-4}{j\pi} ((-1)^j + 1) r^j \sin j\theta \]

**Solution:**

\[ a_0 = \frac{1}{\pi} \int_{0}^{2\pi} \cos 0 \, dx = \frac{1}{\pi} x|_{0}^{2\pi} = 1 \]

\[ a_j = \frac{1}{\pi} \int_{0}^{2\pi} \cos jx \, dx = \frac{1}{\pi} \sin jx |_{0}^{2\pi} = 0 \ (j > 0) \]

\[ b_j = \frac{1}{\pi} \int_{0}^{2\pi} \sin jx \, dx \]

\[ = -\frac{1}{\pi} \cos jx |_{0}^{2\pi} \]

\[ = -\frac{1}{\pi} \left( \frac{j^2}{2} - \frac{(-1)^j}{2} \right) = \frac{1}{j\pi} ((-1)^j - 1) \]

Thus, the solution is

\[ w(r, \theta) = \frac{1}{2} + \sum_{j=1}^{\infty} \frac{1}{j\pi} ((-1)^j - 1) r^j \sin j\theta. \]
11. Using Laplace transforms, the solution to

\[ y'' - y = 2019x \]

with initial conditions \( y(0) = y'(0) = 0 \) is

(a) \( y = 2018(\sinh x - 1) \)
(b) \( y = 2019 \sinh x \)
(c) \( y = 2019(\sinh x - x) \)
(d) \( y = 2020 \sinh x \)
(e) \( y = 2020(\sinh x - 1) \)

Solution:
Taking the Laplace transform of both sides gives

\[
p^2L[y] - py(0) - y'(0) - L[y] = \frac{2019}{p^2}
\]

\[
L[y] = \frac{2019}{(p^2 - 1)p^2}
\]

\[
= \frac{2019}{p^2 - 1} - \frac{2019}{p^2}
\]

Taking the inverse Laplace transform gives

\[ y = 2019(\sinh x - x) \]
12. Using Laplace transforms, the solution to the integral equation

\[ y(x) = \delta(x) + e^x - \int_0^x e^{x-t}y(t) \, dt \]

is

(a) \( y(x) = \delta(x) - x \)
(b) \( y(x) = \delta(x) - 1 \)
(c) \( y(x) = \delta(x) \)
(d) \( y(x) = \delta(x) - x^2 \)
(e) \( y(x) = \delta(x) + x \)

Solution:
Applying the Laplace transform to both sides we get

\[ L[y] = L[\delta(x)] + L[e^x] - L[e^x]L[y]. \]

Solving for \( L[y] \) gives

\[ L[y] = \frac{L[\delta(x)] + L[e^x]}{1 + L[e^x]} \]

\[ = \frac{1 + L[e^x]}{1 + L[e^x]} \]

\[ = 1 \]

This has inverse Laplace transform of

\[ y(x) = \delta(x). \]
13. The impulsive response for the differential equation

\[ y'' + y = f(t) \]

is

(a) \( h(t) = \cos t \)
(b) \( h(t) = \sin t \)
(c) \( h(t) = \sin 2t \)
(d) \( h(t) = \cos 2t \)
(e) \( h(t) = \tan t \)

**Solution:**
Recall that for the impulsive response, \( f(t) = \delta(t) \) and \( h(t) \) is given by the inverse Laplace transform of the reciprocal of the auxiliary equation. Thus,

\[
L^{-1} \left[ \frac{1}{p^2 + 1} \right] = \sin t.
\]