This is a timed examination. You are allowed 120 minutes to finish it.
There are 12 questions worth 10 points each.
This exam covers the entire course but with emphasis on Chapters 10 and 11 of the textbook.

No calculators or other devices may be used.
No books or notes other than a single letter-size sheet of notes and formulas are permitted, nor any collaboration.
Read the statement of each problem carefully.
Be sure to ask questions if anything is unclear.
Show all your work for full credit.
Your ability to make your solution clear will be part of your grade.
1. Write the solution $u(r, \theta)$ to the Dirichlet problem $\Delta u = 0$ on the unit disc $D = \{(r, \theta) : r \leq 1\}$, with boundary condition $u(1, \theta) = \sin^2 \theta$, using the Poisson kernel, and then find the value $u(0, 0)$.

**Solution:** The Poisson integral formula for the solution is

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - r^2) \sin^2 \xi}{1 - 2r \cos(\theta - \xi) + r^2} d\xi$$

When $r = 0$, regardless of $\theta$, it reduces to

$$u(0, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin^2 \xi d\xi = \frac{1}{2},$$

since $\sin^2 \xi = \frac{1}{2} - \frac{1}{2} \cos 2\xi$ and the integral of $\cos 2\xi$ over a complete period $-\pi < \xi < \pi$ is zero.
2. Solve the boundary value problem

\[
\begin{align*}
w_{xx} &= w_t, \quad 0 < x < \pi, \ t \geq 0, \\
w(x,0) &= 2\sin(3x), \quad 0 < x < \pi, \\
w(0,t) &= 0, \quad t \geq 0, \\
w(\pi,t) &= 0, \quad t \geq 0.
\end{align*}
\]

**Solution:** Following Section 11.3, use separation of variables to write the solution as

\[
w(x,t) = \sum_{j=1}^{\infty} b_j e^{-j^2 t} \sin(jx)
\]

which, for all sequences \(\{b_j\}\), satisfies \(w(0,t) = w(\pi,t) = 0\) for all \(t \geq 0\). Then determine \(\{b_j : j = 1, 2, \ldots\}\) from the Fourier sine series for \(f\):

\[
b_j = \frac{2}{\pi} \int_0^{\pi} 2\sin(3x) \sin(jx) \, dx = \begin{cases} 2, & j = 3 \\ 0, & j \neq 3. \end{cases}
\]

Thus the solution is \(w(x,t) = 2e^{-9t} \sin(3x)\).
3. Consider the vibrating string problem for \( y(x,t) \) on \( t \geq 0 \) and \( 0 \leq x \leq \pi \):

\[
\frac{\partial^2 y}{\partial t^2} = k^2 \frac{\partial^2 y}{\partial x^2},
\]

with boundary conditions \( y(0,t) = 0 \) and \( y(\pi,t) = 0 \) for all \( t \), initially at rest with \( \frac{\partial y}{\partial t} |_{t=0} = 0 \). Find the solution if the initial position of the string is given by

\[
y(x,0) = \sin x + \frac{1}{5} \sin 5x, \quad 0 \leq x \leq \pi.
\]

**Solution:** Separation of variables \( y(x,t) = u(x)v(t) \) yields two eigenvalue problems:

\[
\frac{u''(x)}{u(x)} = \frac{v''(t)}{k^2 v(t)} = \lambda.
\]

The boundary conditions in \( x \in [0, \pi] \) force the solution \( u(x) = \sin nx \) for integer \( n \), so \( \lambda = -n^2 \). The initially-at-rest condition is that \( v'(0) = 0 \), which forces the solution \( v(t) = \cos nk t \). Using linearity, we get a general solution of the form

\[
y = \sum_{n=1}^{\infty} b_n \sin nx \cos nk t,
\]

with coefficients \( \{b_n : n = 1, 2, \ldots \} \) yet to be determined.

But the initial shape condition implies that

\[
\sin x + \frac{1}{5} \sin 5x = y(x,0) = \sum_{n=1}^{\infty} b_n \sin nx,
\]

from which we conclude, by comparing Fourier sine expansions, that \( b_1 = 1, b_5 = 1/5, \) and \( b_n = 0 \) for \( n \notin \{1, 5\} \) Thus

\[
y(x,t) = \sin x \cos kt + \frac{1}{5} \sin 5x \cos 5kt.
\]
4. Consider the two-dimensional heat equation

\[
\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial w}{\partial t}
\]

for \( w = w(x, y, t) \). Use the method of separation of variables to find the steady-state solution in the infinite half-strip of the \( xy \) plane consisting of \( \{(x, y) : 0 \leq x \leq \pi, y \geq 0\} \) if the following boundary conditions are satisfied:

\[
\begin{align*}
    w(0, y, t) &= w(\pi, y, t) = 0, & t \geq 0, y \geq 0 \\
    w(x, 0, 0) &= 2 \sin 7x, & 0 \leq x \leq \pi \\
    \lim_{y \to \infty} w(x, y, t) &= 0, & t \geq 0, 0 \leq x \leq \pi.
\end{align*}
\]

**Solution:** The steady-state solution has \( \partial w/\partial t = 0 \), so there is no \( t \)-dependence, and the PDE reduces to

\[
\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial w}{\partial t} = 0,
\]

giving Dirichlet’s equation. Now separate variables \( w(x, y) = u(x)v(y) \) to get two eigenvalue problems from the PDE:

\[
\begin{align*}
    u'' &= \lambda u, & u(0) = u(\pi) = 0; \\
    v'' &= -\lambda v, & v(y) \to 0 \text{ as } y \to \infty.
\end{align*}
\]

The first implies that \( \lambda = n^2 \) for some integer \( n \) and that \( u(x) = \sin nx \). The second implies that \( v(y) = e^{-ny} \) and that \( n > 0 \). Combining these gives

\[
w(x, y) = \sum_{n=1}^{\infty} b_n e^{-ny} \sin nx,
\]

where \( b_n \) is the Fourier sine coefficient of the boundary function \( 2 \sin 7x \), found by setting \( y = 0 \) to get \( w(x, 0) = \sum_{n=1}^{\infty} b_n \sin nx = 2 \sin 7x \). But this has the finite sine series \( b_7 = 2 \), with \( b_j = 0 \) for all \( j \neq 7 \). Thus

\[
w(x, y) = 2e^{-7y} \sin 7x
\]
5. Write the second-order system

\[ x'' = x'y + \frac{t^2}{x^2 + y^2} \quad y'' = y'x + \frac{1 - t^2}{x^2 + y^2} \]

as an equivalent system of first-order equations.

**Solution:** Let \( x_0 = x(t), \ x_1 = x', \ y_0 = y(t), \) and \( y_1 = y'. \) The equivalent system is then

\[
\begin{align*}
    x_0' &= x_1 \\
    x_1' &= x_1y_0 + \frac{t^2}{x_0^2 + y_0^2} \\
    y_0' &= y_1 \\
    y_1' &= y_1x_0 + \frac{1 - t^2}{x_0^2 + y_0^2}
\end{align*}
\]
6. Show that \((x_1, y_1) = (e^{-t}, e^{-t})\) and \((x_2, y_2) = (e^{3t}, -e^{3t})\) are solutions to the homogeneous first-order linear system

\[
\begin{align*}
x' &= x - 2y \\
y' &= -2x + y
\end{align*}
\]

and that they are linearly independent on every closed interval of \(t\).

**Solution:** Check by differentiation and substitution:

\[
\begin{align*}
x_1' &= -e^{-t} = e^{-t} - 2e^{-t} = x_1 - 2y_1; \\
y_1' &= -e^{-t} = -2e^{-t} + e^{-t} = -2x_1 + y_1,
\end{align*}
\]

and likewise

\[
\begin{align*}
x_2' &= 3e^{3t} = e^{3t} - 2(e^{3t}) = x_2 + 2y_2; \\
y_2' &= -3e^{3t} = -2e^{3t} + (-e^{-3t}) = -2x_2 + y_2.
\end{align*}
\]

For linear independence, check the Wronskian:

\[
W(t) = \det \begin{pmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{pmatrix} = \det \begin{pmatrix} e^{-t} & e^{3t} \\ e^{-t} & -e^{3t} \end{pmatrix} = -2e^{2t} < 0, \quad \text{all } t.
\]

Thus \(W(t)\) is never zero for any \(t\). Conclude that \((x_1, y_1)\) and \((x_2, y_2)\) are linearly independent on every closed interval of \(t\).
7. Consider the Volterra predator-prey equation for \( x(t) \) and \( y(t) \) on \( t \geq 0 \):

\[
x' = x - xy; \quad y' = -y + xy,
\]

Find the steady-state solution and compute \( x'' \) and \( y'' \) in terms of \( x \) and \( y \).

**Solution:** To find the steady-state solution, put \( x' = 0 \) and \( y' = 0 \) and solve to get

\[
x(t) = 1, \quad y(t) = 1,
\]

both independent of time.

Differentiate the Volterra system, substitute, and simplify:

\[
x'' = x' - x'y - xy' = x - xy - (x - xy)y - x(-y + xy) = x - xy + xy^2 - x^2 y
\]

and

\[
y'' = -y' + x'y + xy' = y - xy + (x - xy)y + x(-y + xy) = y - xy - xy^2 + x^2 y
\]
8. Solve the initial value problem $y' - y = e^{2x}$, $y(0) = 2$, using the Laplace transform.

**Solution:** Transforming both sides gives $L[y' + y](p) = L[e^{2x}](p)$, which evaluates to

$$(p - 1)L[y](p) - y(0) = \frac{1}{p - 2}.$$ 

But then

$$L[y](p) = \frac{2}{p - 1} + \frac{1}{(p - 1)(p - 2)} = \frac{2}{p - 1} + \left( \frac{1}{p - 2} - \frac{1}{p - 1} \right) = \frac{1}{p - 1} + \frac{1}{p - 2},$$

after adding $y(0) = 2$ to both sides, dividing by $p - 1$, and expanding into partial fractions and simplifying. This gives

$$y(x) = L^{-1} \left[ \frac{1}{p - 1} + \frac{1}{p - 2} \right] (x) = e^{-x} + e^{2x}$$

using linearity and the precomputed Laplace transform $L[e^{ax}](p) = 1/(p - a)$. 
9. Solve the initial value problem $y' - y = e^{2x}$, $y(0) = 0$, using an integrating factor.

**Solution:** This is a first-order linear equation in the form $y' + py = q$ with $p(x) = -1$ and $q(x) = e^{2x}$. The integrating factor is

$$
\mu(x) = \exp \int p(x) \, dx = e^{-x}
$$

and the general solution is

$$
y(x) = \frac{1}{\mu} \int \mu(x)q(x) \, dx = e^x \int e^{-x} \, dx = e^x [e^x + C] = e^{2x} + Ce^x.
$$

Solve for $C = -1$ from the initial condition $y(0) = 0 = 1 + C$ to get

$$
y(t) = e^{2x} - e^x.
$$
10. Find the Fourier series for the function \( g \) defined by

\[
g(x) = \begin{cases} 
-1, & -\pi \leq x < 0, \\
1, & 0 \leq x < \pi.
\end{cases}
\]

**Solution:** First note that \( g \) is an odd function. Hence the Fourier series for \( g \) has no cosine components, so \( a_j = 0 \) for all \( j \geq 1 \), and its sine components are

\[
b_j = \frac{2}{\pi} \int_0^\pi \sin jx \, dx = \frac{2}{\pi} \left[ \frac{-\cos jx}{j} \right]_0^\pi = \frac{2}{\pi} \left[ \frac{-\cos j\pi + 1}{j} \right] = \begin{cases} 
\frac{4}{j\pi}, & \text{if } j \text{ is odd,} \\
0, & \text{if } j \text{ is even,}
\end{cases}
\]

for \( j = 1, 2, 3, \ldots \). Thus,

\[
g(x) = \sum_{\text{odd } j \geq 1} \frac{4}{j\pi} \sin jx = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin(2k+1)x.
\]

Here we write odd \( j \geq 1 \) as \( j = 2k + 1 \) for \( k \geq 0 \).
11. Explain why \( x = 0 \) is an ordinary point for the differential equation \( y' - y = e^{2x} \), and then find the recursion for \( \{a_j : j = 0, 1, 2, \ldots \} \) in the power series solution

\[
y(x) = \sum_{j=0}^{\infty} a_j x^j.
\]

(It is not necessary to solve the recursion.)

**Solution:** Write the equation as \( y' + p(x)y = q(x) \) and observe that the coefficient functions \( p(x) = -1 \) and \( q(x) = e^{2x} \) are analytic in an open interval around 0. Hence \( x = 0 \) is an ordinary point by definition.

The power series for \( y' - y \) is obtained by differentiating term by term:

\[
y'(x) - y(x) = \sum_{j=0}^{\infty} ja_j x^{j-1} - \sum_{j=0}^{\infty} a_j x^j = \sum_{j=0}^{\infty} [(j+1)a_{j+1} - a_j] x^j,
\]
after reindexing and combining terms in \( x^j \). This must equal the power series

\[
e^{2x} = 1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \cdots = \sum_{j=0}^{\infty} \frac{2^j}{j!} x^j,
\]

so the coefficients of \( x^j \) must be equal in both series, so

\[
(j + 1)a_{j+1} - a_j = \frac{2^j}{j!}, \quad \Rightarrow \quad a_{j+1} = \frac{a_j}{j+1} + \frac{2^j}{(j+1)!}
\]

for \( j = 0, 1, 2, \ldots \).
12. The ODE

\[ y^{(6)} + 4y^{(5)} + 3y^{(4)} - 10y^{(3)} - 26y'' - 24y' - 8y = 0 \]

has characteristic equation

\[ r^6 + 4r^5 + 3r^4 - 10r^3 - 26r^2 - 24r - 8 = (r + 1)^2(r^2 + 2r + 2)(r^2 - 4) = 0. \]

Find the general solution.

**Solution:**

\[ y(x) = A_1e^{-x} + A_2xe^{-x} + B_1e^{-x} \cos x + B_2e^{-x} \sin x + Ce^{2x} + De^{-2x}. \]

corresponding to the repeated real root \(-1\) \((A_1e^{-x} + A_2xe^{-x})\), complex conjugate roots \(-1 \pm i\) \((B_1e^{-x} \cos x + B_2e^{-x} \sin x)\), and distinct real roots 2 \((Ce^{2x})\) and \(-2\) \((De^{-2x})\).