Math 233
Krantz

Spring, 2017
May 4, 2017

Final Exam

General Instructions: Read the statement of each problem carefully. Do only what is requested—nothing more and nothing less. Of course you need not show any work for the multiple choice or the TRUE/FALSE questions. For the questions that require a written answer, provide a *complete solution.* If you only write the answer then you will not get full credit.

Be sure to ask questions if anything is unclear. This exam is worth 200 points.

(15 points) **1.** Calculate the tangent plane to the graph of the function $f(x,y) = y^2 - 2xy$ at the point $(2,1,-3)$.

(a) $2x + 2y + z = 3$

(b) $-2x - 2y + z = 3$

(c) $-2x + 2y - z = 3$

(d) $-2x + 2y + z = -3$

(e) $2x + 2y + 2z = 3$

$\underset{=}{n} = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 \rangle = \langle -2y, 2y - 2x, -1 \rangle$

At $(2,1,-3)$, $\underset{=}{n} = \langle -2, -2, -1 \rangle$.

Plane is

$\underset{=}{n} \cdot (\overrightarrow{X} - P) = 0$

or $\langle -2, -2, -1 \rangle \cdot \langle x-2, y-1, z+3 \rangle = 0$

$-2x + 4 - 2y + 2 - z - 3 = 0$

$-2x - 2y - z = -3$

$2x + 2y + z = 3$

(15 points) **2.** Locate and identify the local maxima and local minima and saddle points of the function $f(x,y) = x^3 + 3y^2 - 3x^2 - 6y + 2$.

$\nabla f = \langle 3x^2 - 6x, 6y - 6 \rangle = 0$

$3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0$

$6y - 6 = 0 \Rightarrow y = 1$

Critical points are $(0,1), (2,1)$.

(a) $(0,1)$ is a local min, $(2,1)$ is a local max.

(b) $(0,1)$ is a saddle, $(2,1)$ is a saddle.

(c) $(0,1)$ is a saddle, $(2,1)$ is a local min.

(d) $(0,1)$ is a local max, $(-2,1)$ is a local min.

(e) $(0,-1)$ is a saddle, $(-2,1)$ is a local min.

$\mathcal{D} = \det\begin{pmatrix} 6x-6 & 0 \\ 0 & 6 \end{pmatrix} = 36x - 36$

$\mathcal{D}(0,1) = -36$ so $(0,1)$ is a saddle

$\mathcal{D}(2,1) = 36$, $\partial_{xx} f(2,1) > 0$, $\partial_{yy} f(2,1) > 0$ so $(2,1)$ is a local minimum

(15 points) **3.** Find the extrema of the function $f(x,y) = x^2 - y$ subject to the constraint $g(x,y) = x^2 + y^2 = 4$.

(a) $(\sqrt{15}/2, -1/2)$ is a minimum, $(-\sqrt{15}/2, -1/2)$ is a maximum, $(0,-2)$ is a minumum, $(0,2)$ is not an extremum.

(b) $(\sqrt{15}/2, -1/2)$ is a minimum, $(-\sqrt{15}/2, -1/2)$ is a minimum, $(0,2)$ is a saddle, $(0,-2)$ is not an extremum.

(c) $(\sqrt{15}/2, -1/2)$ is a minimum, $(-\sqrt{15}/2, -1/2)$ is a maximum, $(0,2)$ is a minumum, $(0,-2)$ is a maximum.

(d) $(\sqrt{15}/2, -1/2)$ is a minimum, $(-\sqrt{15}/2, -1/2)$ is a maximum, $(0,2)$ is a minumum, $(0,-2)$ is not an extremum.

(e) $(\sqrt{15}/2, -1/2)$ is a maximum, $(-\sqrt{15}/2, -1/2)$ is a maximum, $(0,2)$ is a minumum, $(0,-2)$ is not an extremum.

$\nabla f = \lambda \nabla g \Rightarrow \langle 2x, -1\rangle = \lambda \langle 2x, 2y\rangle$

$2x = \lambda 2x \Rightarrow 2x(1-\lambda) = 0$

$-1 = \lambda 2y \qquad x^2+y^2=4$

$x = 0 \Rightarrow y = \pm 2 \qquad (0,2),(0,-2)$ are critical points.

$\lambda = 1 \Rightarrow y = -1/2$ so $x^2+(-1/2)^2 = 4 \Rightarrow x^2 + 1/4 = \frac{16}{4}$

$\Rightarrow x = \pm \sqrt{15}/2$. Crit. pts. are $\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right), \left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$.

$f(0,2) = -2$, $f(0,-2) = 2$, $f\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right) = \frac{15}{4} + \frac{1}{2} = \frac{17}{4}$

$f\left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right) = \frac{15}{4} + \frac{1}{2} = \frac{17}{4}$.

So $\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right), \left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$ are maxima.

$(0,2)$ is a minimum

$(0,-2)$ is not a critical point.

(15 points) **4.** The vector field $\mathbf{F}(x,y) = \langle \cos y - y^2, -x\sin y - 2xy\rangle$ is known to be conservative. Find a potential function u.

(a) $u(x,y) = y\cos x - y^2x + C$

(b) $u(x,y) = x\cos y - x^2y + C$

(c) $u(x,y) = x\cos y - yx + C$

(d) $u(x,y) = x\cos y - y^2x + C$

(e) $u(x,y) = x\cos x - y^2x + C$

$\frac{\partial u}{\partial x} = \cos y - y^2$

$\frac{\partial u}{\partial y} = -x\sin y - 2xy$

So $u = x\cos y - xy^2 + \varphi(y)$

$-x\sin y - 2xy = \frac{\partial u}{\partial y} = -x\sin y - 2xy + \varphi'(y)$

Thus $\varphi'(y) = 0$

$\varphi(y) = C$.

We conclude that $u(x,y) = x\cos y - xy^2 + C$.

(20 points) **5.** Reverse the order of integration in order to evaluate the following double integral:

$$\int_0^1 \int_x^1 \frac{e^y}{y}\, dy dx\,.$$

(a) $2e-1$

(b) $e+1$

(c) $e-1$

(d) $-e+1$

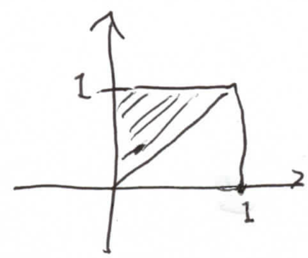
(e) $e-2$

$$\int_0^1 \int_0^y \frac{e^y}{y}\, dx\, dy$$

$$= \int_0^1 \frac{e^y}{y} x \Big]_{x=0}^{x=y} dy$$

$$= \int_0^1 \frac{e^y}{y}\cdot y - \frac{e^y}{y}\cdot 0\, dy = \int_0^1 e^y\, dy$$

$$= e^y \Big]_{y=0}^{y=1} = e^1 - e^0 = e - 1.$$



(15 points) **6.** Calculate the volume of the solid that lies below the graph of $f(x, y) = 1+3x^2$ and over the region in the xy-plane bounded by $y = x^2 - 4$ and $y = -x^2 + 4$.

(a) 1081/15

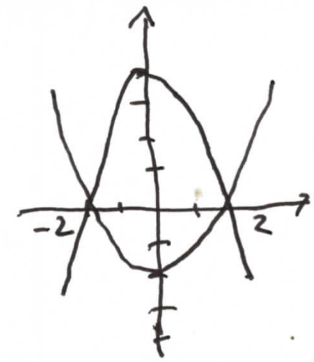
(b) 1088/15

(c) 1082/15

(d) 1041/15

(e) 1110/15

$$\int_{-2}^{2}\int_{x^2-4}^{-x^2+4} 1+3x^2\,dy\,dx = \int_{-2}^{2} y+3x^2y\Big]_{y=x^2-4}^{y=-x^2+4} dx$$



$$= \int_{-2}^{2}\left[(-x^2+4)+3x^2(-x^2+4)\right]-\left[(x^2-4)+3x^2(x^2-4)\right]dx$$

$$= \int_{-2}^{2} -6x^4+22x^2+8\,dx = -\frac{6}{5}x^5+\frac{22}{3}x^3+8x\Big]_{x=-2}^{x=2}$$

$$= \left(-\frac{6\cdot 32}{5}+\frac{22\cdot 8}{3}+16\right)-\left(\frac{6\cdot 32}{5}-\frac{22\cdot 8}{3}-16\right) = \frac{1088}{15}.$$

(15 points) **7.** What are the parametric equations of the normal line to the surface $4x^2+8y^2+z^2=13$ at the point $(1,1,1)$?

$g(x,y,z) = 4x^2+8y^2+z^2=13$

$\nabla g = \langle 8x, 16y, 2z\rangle$ is normal.

At $(1,1,1)$,

$\underline{n} = \langle 8, 16, 2\rangle.$

Parametric equations are

$x = 1+8t$

$y = 1+16t$

$z = 1+2t$

(a)

$$x = 1+8t \quad y = 1+16t \quad z = 1+2t$$

(b)

$$x = 1-8t \quad y = 1+16t \quad z = 1+2t$$

(c)

$$x = 1+8t \quad y = 1-16t \quad z = 1+2t$$

(d)

$$x = 1+16t \quad y = 1+8t \quad z = 1+2t$$

(e)

$$x = 1+2t \quad y = 1+8t \quad z = 1+16t$$

(15 points) **8.** Calculate the curl of the vector field $\mathbf{F}) = y^2z\mathbf{i} + 2xyz\mathbf{j} + \overset{x}{\cancel{z}}y^2\mathbf{k}$.

(a) $\mathbf{i} - \mathbf{j} + \mathbf{k}$

(b) $\mathbf{i} + 0\mathbf{j} + \mathbf{k}$

(c) $0\mathbf{i} + \mathbf{j} + 0\mathbf{k}$

(d) $0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$

(e) $0\mathbf{i} + 0\mathbf{j} - \mathbf{k}$

$$\text{curl} = \det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ y^2z & 2xyz & xy^2 \end{pmatrix}$$

$$= \underline{i}(2yx - 2yx) - \underline{j}(y^2 - y^2) + \underline{k}(2yz - 2yz) = 0\underline{i} + 0\underline{j} + 0\underline{k}.$$

(15 points) **9.** Determine whether the vector field $\mathbf{F}(x, y) = (x^2 - y^2)\mathbf{i} + (2xy - y^2)\mathbf{j}$ is closed. If it is, then find a potential function.

(a) It is closed and the potential function is $u(x, y) = x^2y - y^2x$.

(b) It is closed and the potential function is $u(x, y) = -x^2y + y^2x$.

(c) It is closed and the potential function is $u(x, y) = -x^2y - y^2x$.

(d) It is closed and the potential function is $u(x, y) = x^2y + y^2x$.

(e) It is not closed.

$\frac{\partial}{\partial y}(x^2 - y^2) = -2y$

$\frac{\partial}{\partial x}(2xy - y^2) = 2y$

Unequal so not closed.

20 points) **10.** Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F}(x, y) = xy\mathbf{i} + xy^2\mathbf{j}$ and $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$, $1 \le t \le 2$.

(a) 91/5

(b) 93/7

(c) 93/5

(d) 73/5

(e) 39/5

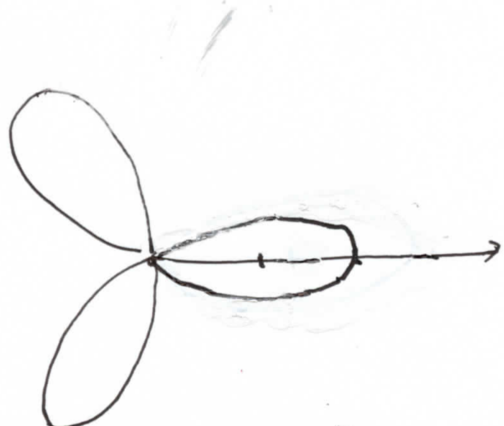
$$\int_C \underline{F} \cdot d\underline{r} = \int_1^2 \underline{F}(\underline{r}(t)) \cdot \underline{r}'(t)\, dt$$

$$= \int_1^2 (t^3 \underline{i} + t^4 \underline{j}) \cdot (2t\underline{i} + 1\underline{j})\, dt$$

$$= \int_1^2 2t^4 + t^4\, dt = \int_1^2 3t^4\, dt$$

$$= \frac{3t^5}{5}\Big]_1^2 = \frac{96}{5} - \frac{3}{5} = \frac{93}{5}.$$

20 points) **11.** Sketch the curve $r = 2\cos(3\theta)$ in polar coordinates.



$0 \le \theta \le \frac{\pi}{6} \Rightarrow 2 \to 2\cos 3\theta \to 0$

$\frac{\pi}{6} \le \theta \le \frac{\pi}{3} \Rightarrow 0 \to 2\cos 3\theta \to -2$

$\frac{\pi}{3} \le \theta \le \frac{\pi}{2} \Rightarrow -2 \le 2\cos 3\theta \le 0$

$\frac{\pi}{2} \le \theta \le \frac{2\pi}{3} \Rightarrow 0 \le 2\cos 3\theta \le 2$

$\frac{2\pi}{3} \le \theta \le \frac{5\pi}{6} \Rightarrow 2 \to 2\cos 3\theta \to 0$

$\frac{5\pi}{6} \le \theta \le \pi \Rightarrow 0 \to 2\cos\theta \to -2$

$\pi \le \theta \le \frac{7\pi}{6} \Rightarrow -2 \le 2\cos\theta \le 0$

We are now retracing the same curves.

(20 points) **12.** Use Green's theorem to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ by integrating over the region $\mathcal{R}$ that C bounds. Here $\mathbf{F}(x, y) = 3y\mathbf{i} - 4x\mathbf{j}$ and C is given by $\mathbf{r}(t) = 3 \sin t\mathbf{i} - 4 \cos t\mathbf{j}$, $0 \le t \le 2\pi$.

Of course C bounds the ellipse $16x^2 + 9y^2 \le 144$,

So we see that $-3 \le x \le 3$ and

$$9y^2 \le 144 - 16x^2$$

$$|3y| \le \sqrt{144 - 16x^2}$$

$$-\frac{1}{3}\sqrt{144-16x^2} \le y \le \frac{1}{3}\sqrt{144-16x^2}$$

Thus

$$\oint_C \underline{F} \cdot d\underline{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \int_{-3}^{3} \int_{-\frac{1}{3}\sqrt{144-16x^2}}^{\frac{1}{3}\sqrt{144-16x^2}} -4-3 \quad dy\, dx$$

$$= \int_{-3}^{3} \int_{-\frac{1}{3}\sqrt{144-16x^2}}^{\sqrt{144-16x^2}} -7 \, dy\, dx$$

$= -7 \cdot (\text{area of ellipse})$

$= -7 \cdot \pi \cdot 3 \cdot 4 = -84\pi.$