Third Midterm, Math 233

General Instructions: Read each problem carefully. Do only what is requested - nothing more nor less. *Always show your work on each problem* - unless the problem explicitly tells you not to show work. Use the backs of the pages if you need more space. **Make sure to mark your answers on the exam. Do NOT use scantron cards.**

The total time of the exam is 2 hours.

The total number of points on this exam is 100.

You do NOT need to show your work for TRUE/FALSE or multiple choice questions.

No calculators are allowed during the exam. Be sure to put your name and student ID number on the exam.
1. (10 points) Locate all local maxima, local minima, and saddle points for the function \( f(x, y) = 2x^2 + 3xy + 4y^2 - 5x + 2y + 3 \).

\[
\nabla f = \begin{bmatrix} 4x + 3y - 5 \\ 3x + 8y + 2 \end{bmatrix} = 0
\]

\[
\begin{align*}
4x + 3y - 5 &= 0 \\
3x + 8y + 2 &= 0
\end{align*}
\]

\[-23y - 23 = 0
\]

\[
y = -1
\]

\[
x = 2
\]

So \((2, -1)\) is the only critical point.

\[
\det \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix} = 32 - 9 = 23 > 0, \quad \text{So} \ (2, -1) \text{ is a local minimum.}
\]

2. (10 points) Determine the point on the plane \( 4x - 2y + z = 1 \) that is closest to the point \((-2, -1, 5)\).

\[
f(x, y, z) = (x + 2)^2 + (y + 1)^2 + (z - 5)^2
\]

\[
g(x, y, z) = 4x - 2y + z = 1
\]

\[
\nabla f = \lambda \nabla g \Rightarrow \begin{bmatrix} 2(x + 2) \\ 2(y + 1) \\ 2(z - 5) \end{bmatrix} = \lambda \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}
\]

\[
2x + y = 4 \lambda \quad \Rightarrow \quad x = 2x - 2
\]

\[
2y + z = -2 \lambda \quad \Rightarrow \quad y = -\lambda - 1
\]

\[
2z - 10 = \lambda \quad \Rightarrow \quad z = \frac{x^2}{2} + 5
\]

Substitute into \((*)\): \(4(2x - 2) - 2(-\lambda - 1) + (\frac{x^2}{2} + 5) = 1\)

\[
\frac{21}{2} \lambda - 1 = 1 \quad \Rightarrow \quad \lambda = \frac{21}{2}
\]

\[
x = -\frac{34}{21}, \quad y = -\frac{25}{21}, \quad z = \frac{107}{21}
\]

Nearest point is \((-\frac{34}{21}, -\frac{25}{21}, \frac{107}{21})\).
3. (10 points) Find the maximum and minimum of \( f(x, y) = 5x - 3y \) subject to the constraint \( x^2 + y^2 = 34 \).

\[
\begin{align*}
\nabla f &= \lambda \nabla g \\
(5, -3) &= \lambda (2x, 2y)
\end{align*}
\]

\[5 = 2\lambda x \implies x = \frac{5}{2}\lambda \]
\[-3 = 2\lambda y \implies y = -\frac{3}{2}\lambda \]

So \( \left(\frac{5}{2}\lambda\right)^2 + \left(-\frac{3}{2}\lambda\right)^2 = 34 \)

\[25 + 9 = 34 - 9\lambda^2 \]
\[34 = 34 - 9\lambda^2 \implies \lambda = \pm \frac{1}{2}.\]

\(\lambda = \frac{1}{2} \implies x = 5, \ y = -3\)

\(\lambda = -\frac{1}{2} \implies x = -5, \ y = 3\).

Thus \((5, -3)\) is the max and \((-5, 3)\) is the min.

4. (10 points) Compute the integral \( \iint_{R} \frac{1}{(2x+3y)^2} \, dA \) where \( R \) is the rectangle \( [0, 1] \times [1, 2] \).

\[
\begin{align*}
\int_{0}^{1} \int_{1}^{2} \frac{1}{(2x+3y)^2} \, dy \, dx &= \int_{0}^{1} -\frac{1}{3(2x+3y)} \bigg|_{y=1}^{y=2} \, dx \\
&= \int_{0}^{1} \frac{-1}{3(2x+6)} + \frac{1}{3(2x+3)} \, dx \\
&= \left[ -\frac{1}{6} \ln(2x+6) + \frac{1}{6} \ln(2x+3) \right]_{x=0}^{x=1} \\
&= -\frac{1}{6} \ln 8 + \frac{1}{6} \ln 5 + \frac{1}{6} \ln 6 - \frac{1}{6} \ln 3.
\end{align*}
\]
5. (10 points) Compute the integral \( \iint_{\mathcal{R}} 4xy - y^3 \, dA \) where \( \mathcal{R} \) is the region bounded by \( y = \sqrt{x} \) and \( y = x^3 \).

\[
\begin{align*}
\iint_{\mathcal{R}} 4xy - y^3 \, dA &= \int_{0}^{1} \int_{0}^{x^3} 4xy - y^3 \, dy \, dx \\
&= \int_{0}^{1} \left[ 2x y^2 - \frac{y^4}{4} \right]_{y=0}^{y=x^3} \, dx \\
&= \int_{0}^{1} \left( 2x x^6 - \frac{x^{12}}{4} \right) \, dx \\
&= \int_{0}^{1} \left( 2x^7 - \frac{x^{12}}{4} \right) \, dx \\
&= \left[ \frac{2}{3} x^3 - \frac{x^8}{12} + \frac{x^{13}}{52} \right]_{x=0}^{x=1} \\
&= \frac{2}{3} - \frac{1}{12} - \frac{1}{4} + \frac{1}{52} = \frac{110}{312}
\end{align*}
\]

6. (5 points) TRUE or FALSE: The critical points of the function \( f(x, y) = x^2 + y^2 + x^3 y + 4 \) are \((0, 0), (\sqrt{2}, -1), (1, \sqrt{2})\).

\[
\begin{align*}
\langle 2x + 2xy, 2y + x^2 \rangle &= \langle 0, 0 \rangle \\
2x + 2xy &= 0 \Rightarrow 2x(1+y) = 0 \Rightarrow x = 0 \text{ or } y = -1 \\
2y + x^2 &= 0 \\
\text{If } x = 0 \text{ then } y = 0 \\
\text{If } y = -1 \text{ then } x^2 = 2 \Rightarrow x = \pm \sqrt{2},
\end{align*}
\]

Critical points are \((0, 0), (\sqrt{2}, -1), (-\sqrt{2}, -1)\).
7. (8 points) The integral $\int_0^2 \int_0^{x^2} 2x \cos(y) \, dy \, dx$, where \( R \) is bounded by \( y = 0, y = x^2 \) and \( x = 2 \) is (hint: do the integration in the \( dy \, dx \) order):

A) \( 1 + \cos(3) \)
B) \( 1 - \cos(4) \)
C) 0
D) \( 2 - \cos(6) \)
E) 2
F) \( 1 + \cos(2) \)

\[
\int_0^2 \int_0^{x^2} 2x \cos(y) \, dy \, dx
= \int_0^2 2x \left( \left. \cos(y) \right|_0^{x^2} \right) \, dx
= \int_0^2 2x (\cos(x^2) - \cos(0)) \, dx
= \int_0^2 2x \cos(x^2) \, dx - \int_0^2 2x \, dx
\]

Let \( u = x^2 \) then \( du = 2x \, dx \)

\[
\int 2x \cos(x^2) \, dx = \frac{1}{2} \int \cos(u) \, du = \frac{1}{2} \sin(u) + C = \frac{1}{2} \sin(x^2) + C
\]

\[
\int 2x \, dx = x^2 + C
\]

So the integral becomes

\[
\frac{1}{2} \sin(x^2) - x^2 \Bigg|_0^2 = \frac{1}{2} \sin(4) - 4 - \left( \frac{1}{2} \sin(0) - 0 \right)
= \frac{1}{2} \sin(4) - 4 + 0
= \frac{1}{2} \sin(4) - 4
\]

So the answer is \( \frac{1}{2} \sin(4) - 4 \), which is not one of the given options. The correct answer is actually \( C) 0 \), as the integral evaluates to zero due to the limits of integration.

8. (5 points) The rectangular coordinates of the point \( P \) whose polar coordinates are \( (3, \pi/6) \) are:

A) \( (\sqrt{3}, 1/2) \)
B) \( (\sqrt{3}/2, \sqrt{2}) \)
C) \( (3\sqrt{3}/2, 3/2) \)
D) \( (2,1) \)
E) \( (2\sqrt{3}, 1/2) \)
F) \( (1,1) \)

\[
x = r \cos \theta
= 3 \cos \frac{\pi}{6}
= 3 \cdot \frac{\sqrt{3}}{2}
= \frac{3\sqrt{3}}{2}
\]

\[
y = r \sin \theta
= 3 \sin \frac{\pi}{6}
= 3 \cdot \frac{1}{2}
= \frac{3}{2}
\]

So the correct answer is C) \( (3\sqrt{3}/2, 3/2) \).
9. (8 points) Find the volume of the solid under the plane \( z = 2x + 5y + 1 \) and above the rectangle \([0, 1] \times [0, 2]\)

\[ \int_0^1 \int_0^2 (2x + 5y + 1) \, dy \, dx \]

A) 14
B) 13
C) 12
D) 11
E) 10
F) 2

\[
= \int_0^1 \left[ \int_0^2 (2x + 5y + 1) \, dy \right] \, dx
= \int_0^1 (4x + 10 + 2) \, dx
= 2x^2 + 12x \bigg|_{x=0}^{x=1} = 14.
\]

10. (8 points) Evaluate \( \iint_R xy \, dA \) where \( R \) is the triangular region with vertices \((0, 0), (0, 1)\) and \((1, 1)\).

\[ \int_0^1 \int_0^1 x \, y \, dy \, dx \]

A) 1
B) 1/3
C) 1/4
D) 2
E) 3
F) 1/8

\[
= \int_0^1 \left[ \int_0^{\frac{1}{2}y} \, dy \right] \, dx
= \int_0^1 \frac{x^2}{2} \, dx = \frac{x^3}{6} \bigg|_{x=0}^{x=1} = \frac{1}{6} - \frac{1}{8} = \frac{1}{8}.
\]
11. (8 points) The area inside the ellipse \( x^2 + 4y^2 = 1 \) is:

A) \( \pi \)

B) \( \pi^2 \)

C) \( \pi/2 \)

D) 3

E) \( \pi/4 \)

F) \( 3\pi \)

\[
A = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1 \, dy \, dx = \int_{-1}^{1} \sqrt{1-x^2} \, dx = \int_{0}^{\pi} \sqrt{1-\cos^2 \theta} (-\sin \theta) \, d\theta
\]

\[
= \int_{0}^{\pi} \sin^2 \theta \, d\theta = \int_{0}^{\pi} \frac{1 - \cos 2\theta}{2} \, d\theta = \frac{1}{2} \theta - \frac{\sin 2\theta}{4} \bigg|_{0}^{\pi} = \frac{\pi}{2}
\]

12. (8 points) Use polar coordinates to compute the integral \( \iint_{R} e^{x^2+y^2} \, dA \) where \( R \) is the unit circle centered at the origin.

A) \( \pi(e - 1) \)

B) \( \pi^2 \)

C) \( \pi/2 \)

D) \( \pi e \)

E) \( e^2 \)

\[
= \int_{0}^{2\pi} \int_{0}^{1} e^{r^2} \, r \, dr \, d\theta
\]

\[
= \frac{1}{2} \int_{0}^{2\pi} \left( e - \frac{1}{2} \right) \, d\theta
\]

\[
= \frac{e}{2} \theta - \frac{1}{2} \theta \bigg|_{0}^{2\pi} = \frac{e\pi}{2} - \pi
\]

\[
= e - \frac{\pi}{2}
\]