Second Midterm: Makeup Exam

General Instructions: Read each problem carefully. Do only what is requested—nothing more nor less. Always show your work on each problem—unless the problem explicitly tells you not to show work. You will not get full credit for just writing down the answer. The point value of each problem is shown. Use the backs of the pages if you need more space to solve a problem.

The total time on this exam is two hours.

The total points on this exam is 100.

Ask questions if anything is unclear.

There are several TRUE/FALSE and multiple choice questions. For those problems you need not show any work.

No calculators are allowed during this exam. Be sure to put your name and student ID number on the exam.

1. (10 points) Find the center of curvature, the radius of curvature, and the osculating plane for the curve \( r(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + t \mathbf{k} \) at the point \((0, -1, \pi)\).

\[
\begin{align*}
\mathbf{r}'(t) &= \cos t \mathbf{i} - \sin t \mathbf{j} + \mathbf{k} \\
|| \mathbf{r}'(t) || &= \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2} \\
\mathbf{T}(t) &= \frac{\cos t \mathbf{i} - \sin t \mathbf{j}}{\sqrt{2}} + \frac{1}{\sqrt{2}} \mathbf{k} \\
\mathbf{T}'(t) &= -\sin t \mathbf{i} - \cos t \mathbf{j} + 0 \mathbf{k} \\
|| \mathbf{T}'(t) || &= \sqrt{\sin^2 t + \cos^2 t} = \frac{1}{\sqrt{2}} \\
\mathbf{N}(t) &= -\sin t \mathbf{i} - \cos t \mathbf{j} + 0 \mathbf{k} \\
\mathbf{r}''(t) &= -\sin t \mathbf{i} - \cos t \mathbf{j} + 0 \mathbf{k}
\end{align*}
\]
The normal component of acceleration for the motion \( r(t) = t^3i - tj + t^2k \) at the point \((1, -1, 1)\) is given by

\[
\mathbf{a} = \frac{\mathbf{a}_N}{k} = \frac{\mathbf{r}''(t) \times \mathbf{r}''(t)}{||\mathbf{r}''(t)||^3}
\]

where \( \mathbf{a}_N \) is the normal component of acceleration. The direction of the normal component of acceleration at the point \((1, -1, 1)\) is given by

\[
\mathbf{b} = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}
\]

Osculating Plane:

\[
\mathbf{b} \cdot (\mathbf{r} - \langle 0, -1, 1 \rangle) = 0 \Rightarrow \langle -1/2, 0, 0 \rangle \times \langle x, y + 1, 2 - t \rangle = 0
\]

2. (8 points) The normal component of acceleration for the motion \( r(t) = t^3i - tj + t^2k \) at the point \((1, -1, 1)\) is given by

\[
\mathbf{r}'(t) = 3t^2i - k + 2ti,
\]

\[
\mathbf{r} = ||\mathbf{r}'(t)|| = \sqrt{9t^4 + 1 + 4t^2}
\]

\[
\frac{dv}{dt} = \frac{1}{2} \left( 9t^4 + 4t^2 + 1 \right)^{1/2} - (36t^3 + 8t)
\]

At \( t = 2 \) this gives

\[
\frac{dv}{dt} = \frac{1}{2} \frac{4}{\sqrt{14}} = \frac{2}{\sqrt{14}}
\]

\[
a = \frac{\mathbf{a}''(t)}{2} = 6t\frac{1}{2} + 2t\frac{1}{2}
\]

\[
||\mathbf{a}|| = \sqrt{36t^2 + 4}
\]

At \( t = 2 \) this is \( ||\mathbf{a}|| = \sqrt{40} \)

\[
\mathbf{v} = \sqrt{11^2 - \left| \frac{dv}{dt} \right|^2} = \sqrt{40 - \frac{22^2}{14}}
\]

\[
= \sqrt{40 - \frac{484}{14}} = \sqrt{\frac{560 - 484}{14}} = \frac{\sqrt{76}}{14} = \frac{\sqrt{38}}{7}
\]

I made a mistake and did not list this answer.
3. (4 points) Write down the integral for the arc length of the curve \( r(t) = \cos t\mathbf{i} - \sin 2t\mathbf{j} + t^2\mathbf{k} \) between \( t = 2 \) and \( t = 4 \). Do not attempt to evaluate this integral.

\[
\int_{2}^{4} \sqrt{\sin^2 t + 4\cos^2 2t + 4t^2} \, dt
\]
4. (6 points) Sketch the graph of the function \( z = f(x, y) = x + y^2 \) on this three-dimensional set of axes. Be sure to exhibit your calculations of level sets, and be sure to label the level sets in your graph.

<table>
<thead>
<tr>
<th>Level Set</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z = -2 )</td>
<td>( x = -y^2 - 2 )</td>
</tr>
<tr>
<td>( z = -1 )</td>
<td>( x = -y^2 - 1 )</td>
</tr>
<tr>
<td>( z = 0 )</td>
<td>( x = -y^2 )</td>
</tr>
<tr>
<td>( z = 1 )</td>
<td>( x = -y^2 + 1 )</td>
</tr>
<tr>
<td>( z = 2 )</td>
<td>( x = -y^2 + 2 )</td>
</tr>
</tbody>
</table>
5. (6 points) Sketch the locus of points (the cylinder) described by the equation $y^2 + 2x^2 = 4$. 

\begin{center}
\begin{tabular}{c|c|c}
<table>
<thead>
<tr>
<th>Level Set</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$-2$</td>
<td>$y^2 + 2x^2 = 4$ ellipse</td>
</tr>
<tr>
<td>$-1$</td>
<td>$y^2 + 2x^2 = 4$</td>
</tr>
<tr>
<td>$0$</td>
<td>$y^2 + 2x^2 = 4$</td>
</tr>
<tr>
<td>$1$</td>
<td>$y^2 + 2x^2 = 4$</td>
</tr>
<tr>
<td>$2$</td>
<td>$y^2 + 2x^2 = 4$</td>
</tr>
</tbody>
</table>
\end{tabular}
\end{center}
6. (6 points) Let \( f(x, y) = y^2 \sin(xy^2) \). Calculate these partial derivatives:

(a) \( \frac{\partial f}{\partial y} = 2y \sin(xy^2) + 2xy^3 \cos(xy^2) \)

(b) \( \frac{\partial^2 f}{\partial x \partial y} = 2y^3 \cos(xy^2) + 2xy^3 \cos(xy^2) - 2xy^5 \sin(xy^2) \\
= 4y^3 \cos(xy^2) - 2xy^5 \sin(xy^2) \)

7. (8 points) The parametric equations for the tangent line to the curve \( \mathbf{r}(t) = t^3 \mathbf{i} - t^2 \mathbf{j} + 2t \mathbf{k} \) at the point \((8, -4, 4)\) are given by (circle your answer)

(a) \[
\begin{align*}
x &= 3 + 3t \\
y &= -8 - 10t \\
z &= 6 + 2t
\end{align*}
\]

(b) \[
\begin{align*}
x &= 4 + 4t \\
y &= -8 - 12t \\
z &= 6 + 3t
\end{align*}
\]
(c) 

\begin{align*}
x &= 4 + 3t \\
y &= -7 - 12t \\
z &= 6 + 2t
\end{align*}

\[ x' = 3t^2 \hat{i} + 2t \hat{j} + 2 \hat{k} \quad \Rightarrow \quad \mathbf{r}'(2) = 12 \hat{i} + 4 \hat{j} + 2 \hat{k} \]

Tangent line:

\begin{align*}
x &= 8 + 12t \\
y &= -4 - 4t \\
z &= 4 + 2t
\end{align*}

8. (8 points) The curvature of the curve \( \mathbf{r}(t) = t^2 \mathbf{i} + t \mathbf{j} - \sin \mathbf{t} \mathbf{k} \) at the point \((0, 0, 0)\) is given by (circle your answer)

(a) 2
(b) 1
(c) 1/2
(d) 1/3

\[ r'(t) = 2t \mathbf{i} + \mathbf{j} - \cos t \mathbf{k} \]
\[ r''(t) = 2 \mathbf{i} + 0 \mathbf{j} + \sin t \mathbf{k} \]
\[ r'(t) \times r''(t) = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t & 0 & \sin t \\ -\cos t & 0 & -1 \end{pmatrix} = \mathbf{i}(\sin t) - \mathbf{j}(2t \sin t + 2 \cos t) + \mathbf{k}(-2) \]
\[ |r'(0) \times r''(0)| = \sqrt{0^2 + 1^2 + 2^2} = \sqrt{5} \]
\[ |r'(0)| = \sqrt{0^2 + 7^2 + 1^2} = \sqrt{50} \]
\[ \kappa(0) = \frac{|r'(0) \times r''(0)|}{|r'(0)|^3} = \frac{\sqrt{5}}{2\sqrt{2}} = \frac{\sqrt{10}}{4} \]
9. (6 points) For the function \( g(x, y) = x^2y - y^2x \) and the direction vector \( \mathbf{u} = \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle \), the value of \( D_ug(1, 1) \) is (circle your answer)

(a) \( -\sqrt{3} \)
(b) \( \sqrt{2} \)
(c) \( -\sqrt{2} \)
(d) \( \sqrt{3} \)

\[ \nabla g(x, y) = \langle 2xy - y^2, x^2 - 2yx \rangle \]
\[ \nabla g(1, 1) = \langle 1, -1 \rangle \]
\[ D_ug(1, 1) = \langle 1, -1 \rangle \cdot \langle -1/\sqrt{2}, 2/\sqrt{2} \rangle \]
\[ \quad = -1/\sqrt{2} - 2/\sqrt{2} = -3/\sqrt{2} = -\sqrt{2} \]

10. (8 points) For the function

\[ f(x, y) = \begin{cases} 
2xy & \text{if } (x, y) \neq 0 \\
2x^2 + y^2 & \text{if } (x, y) = 0
\end{cases} \]
the limit \( \lim_{(x,y) \to (0,0)} f(x, y) \) (circle your answer)

(a) exists
(b) does not exist
(c) is trying to exist
(d) may be forbidden to exist

\[
\begin{align*}
\lim_{t \to 0} f(t, 0) &= \lim_{t \to 0} 0 = 0 \\
\lim_{t \to 0} f(0, t) &= \lim_{t \to 0} 0 = 0 \\
\lim_{t \to 0} f(t, t) &= \lim_{t \to 0} \frac{2}{3} = \frac{2}{3} \\
\therefore \lim_{(x,y) \to (0,0)} f(x, y) \text{ does not exist.}
\end{align*}
\]

11. (8 points) TRUE or FALSE: If \( f(x, y) = x^2 - y^3 \) and if \( x = s^3 - t^2 \) and \( y = s^2 + t^3 \), then

\[
\frac{\partial f}{\partial s} = 4s^3 - 4st^3 - 8s^4 - 10s^5 t^2 - 9s^2 t^4.
\]

Put your answer here:

\[
\begin{align*}
\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \\
&= 2x \cdot 3s^2 - 3y^2 \cdot 2s \\
&= 2(s^3 - t^2) \cdot 3s^2 - 3(s^2 + t^3)^2 \cdot 2s \\
&= 6s^5 - 6t^2 s^2 - 3(s^4 + 2s^2 t^3 + t^6) \cdot 2s \\
&= 6s^5 - 6t^2 s^2 - 6s^5 - 12s^3 t^3 - 6s t^6 \\
\text{This does not equal the suggested answer, so FALSE.}
\end{align*}
\]
12. (8 points) TRUE or FALSE: The tangent plane to the graph of \( z = f(x, y) = x^2 + y \) at the point \( (1, 2, 3) \) is given by

\[ 2x + y - z = 1. \]

Put your answer here:

\[
\begin{array}{c}
\text{True} \\
\text{False}
\end{array}
\]

Normal to graph is \( \langle 2x, 1, -1 \rangle \).
At the given point it is \( \langle 2, 1, -1 \rangle \).
Tangent plane is

\[
\langle 2, 1, -1 \rangle \cdot (x-1, y-2, z-3) = 0
\]

\[ 2x + y - z = 1. \]
13. (8 points) The gradient of the function \( f(x, y) = xy \sin y^2 \) is (circle your answer)

(a) \( (y \sin y^2, x \sin y^2 + 2xy^2 \cos y^2) \)
(b) \( (x \cos y^2 - 2y^2 x \sin y^2, y \cos y^2) \)
(c) \( (y \sin x^2 - 2x^2 y \cos x^2, x \sin x^2) \)
(d) \( (x \cos x^2 - 2y^2 \sin x^2, y \sin x^2) \)

\[ \nabla f = \left< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right> = \left< y \sin y^2, x \sin y^2 + 2xy^2 \cos y^2 \right> \]

14. (6 points) TRUE or FALSE: The domain of the function \( g(x, y) = \sqrt{9 - x^4 - y^2} \) is the set \( S = \{(x, y) : x^4 + y^2 \leq 9\} \). Put your answer here:

\( \square \)

The domain is the set of points where
\[ 9 - x^4 - y^2 \geq 0 \]
\[ x^4 + y^2 \leq 9, \]