Math 233
Krantz

Spring, 2017
March 7, 2017

Second Midterm

Practice

General Instructions: Read the statement of each problem carefully. Do only what is requested—nothing more and nothing less. Of course you need not show any work for the multiple choice or the TRUE/FALSE questions. For the questions that require a written answer, provide a *complete solution.* If you only write the answer then you will not get full credit.

Be sure to ask questions if anything is unclear. This exam is worth 100 points.

(10 points) **1.** A curve in space is given by

$$\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j} - 2t\mathbf{k}\,.$$

Calculate the unit tangent vector at the point $(1, 1, -2)$.

The point $(1,1,-2)$ corresponds to $t = 1$.

$$\mathbf{r}'(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} - 2\,\mathbf{k}$$

$$\mathbf{r}'(1) = 2\,\mathbf{i} + 3\,\mathbf{j} - 2\,\mathbf{k}$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{17}}(2\,\mathbf{i} + 3\,\mathbf{j} - 2\,\mathbf{k})$$

$$= \frac{2}{\sqrt{17}}\,\mathbf{i} + \frac{3}{\sqrt{17}}\,\mathbf{j} - \frac{2}{\sqrt{17}}\,\mathbf{k}\,.$$

(10 points) **2.** What is the tangent line to the curve $\mathbf{r}(t) = (\cos 2t)\mathbf{i} + (\sin 2t)\mathbf{j} + t\mathbf{k}$ at the point $(-1, 0, \pi/2)$?

The point $(-1, 0, \pi/2)$ corresponds to $t = \pi/2$.

$\underline{r}'(t) = -2\sin 2t\,\underline{i} + 2\cos 2t\,\underline{j} + 1\underline{k}$

$\underline{r}'(1) = -2\sin 2\,\underline{i} + 2\cos 2\,\underline{j} + 1\,\underline{k}$

The tangent line is

$$x = -1 + (-2\sin 2)t$$
$$y = 0 + (2\cos 2)t$$
$$z = \frac{\pi}{2} + 1t$$

(10 points) **3.** The length of that portion of the curve $\mathbf{r}(t) = \cos t^2\mathbf{i} + \sin t^2\mathbf{j} + \mathbf{t}^2\mathbf{k}$ between the points $(1, 0, 0)$ and $(-1, 0, \pi)$ is what?

$\underline{r}'(t) = -2t\sin t^2\,\underline{i} + 2t\cos t^2\,\underline{j} + 2t\,\underline{k}$

$\|\underline{r}'(t)\| = \sqrt{4t^2\sin^2 t^2 + 4t^2\cos^2 t^2 + 4t^2}$

$= \sqrt{4t^2 + 4t^2} = t\sqrt{8}$

$(1, 0, 0) \leftrightarrow t = 0$

$(-1, 0, \pi) \leftrightarrow t = \sqrt{\pi}$

$$\text{length} = \int_0^{\sqrt{\pi}} \sqrt{8}\,t\,dt = \frac{\sqrt{8}}{2}t^2\Big]_0^{\sqrt{\pi}} = \frac{\sqrt{8}\,\pi}{2} - 0 = \frac{\sqrt{8}\,\pi}{2} = \sqrt{2}\,\pi.$$

(10 points) **4.** What does Kepler's Third Law say?

The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its elliptical orbit, with the same constant of proportionality for every planet.

(10 points) **5. TRUE** or **FALSE**: If a curve has curvature 0 at each point then it is a line.

If $K = 0$ at each point then $\underline{\underline{T}}' \equiv 0$

So $\underline{\underline{T}}$ is constant. Hence the curve is a line.

(10 points) **6.** The curvature of the curve $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j} + t\mathbf{k}$ at the point $(1, 1, 1)$ is what?

$$\underline{r}'(t) = 2t\,\underline{i} + 3t^2\,\underline{j} + 1\,\underline{k}, \quad \underline{r}''(t) = 2\,\underline{i} + 6t\,\underline{j} + 0\,\underline{k}$$

$$\underline{r}'(1) = 2\,\underline{i} + 3\,\underline{j} + 1\,\underline{k}, \quad \underline{r}''(1) = 2\,\underline{i} + 6\,\underline{j} + 0\,\underline{k}$$

$$\underline{r}'(1) \times \underline{r}''(1) = \det\begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 3 & 1 \\ 2 & 6 & 0 \end{pmatrix} = \underline{i}(-6) - \underline{j}(-2) + \underline{k}\cdot 6$$

$$\|\underline{r}'(1) \times \underline{r}''(1)\| = \sqrt{36+4+36} = \sqrt{76}$$

$$\|\underline{r}'(1)\| = \sqrt{4+9+1} = \sqrt{14}$$

$$\kappa = \frac{\sqrt{76}}{(\sqrt{14})^3}.$$

(10 points) **7.** What is the direction of greatest increase of the function $f(x, y) = x^2 + y^4$ at the point $(1, 1)$?

$$\nabla f = \langle 2x, 4y^3 \rangle$$

$$\nabla f(1,1) = \langle 2, 4 \rangle$$

$$\underline{u} = \frac{\langle 2, 4\rangle}{\sqrt{20}} = \frac{1}{\sqrt{5}}\langle 1, 2\rangle$$

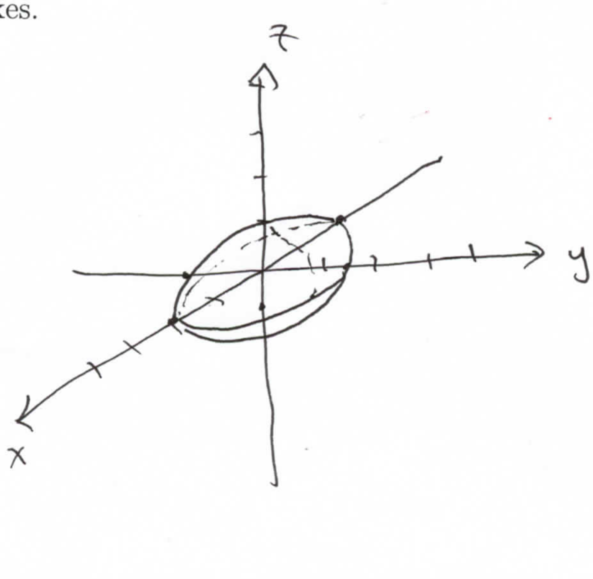
is the direction of greatest increase.

(10 points) **8.** Let $\mathbf{u} = \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle$ be a direction vector. Define $f(x, y) = ye^x$. Calculate $D_{\mathbf{u}}f$.

$$D_{\underline{u}} f = \underline{u} \cdot \nabla f = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \cdot \langle ye^x, e^x \rangle$$

$$= -\frac{1}{\sqrt{2}} ye^x + \frac{1}{\sqrt{2}} e^x.$$

(10 points) **9.** On a set of x-y-z axes, sketch the graph of $x^2+2y^2+4z^2 = 4$. Be sure to show coordinates on your axes.



(10 points) **10.** Let $f(x, y) = ye^{xy}$. Calculate $\partial f/\partial x$ and $\partial f/\partial y$.

$$\frac{\partial f}{\partial x} = y^2 e^{xy}.$$

$$\frac{\partial f}{\partial y} = e^{xy} + xye^{xy}$$