MATH 309

SOLUTIONS TO PRACTICE FINAL

1. If $p, q \in \mathcal{P}_5$, then $p+q \in \mathcal{P}_5$.
If $p \in \mathcal{P}_5$ and $c \in \mathbb{R}$, then $cp \in \mathcal{P}_5$.
Also $0 \in \mathcal{P}_5$.

2. $$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 4 & 6 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 4 & 6 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 is free so $x_3 = \lambda$, $x_2 = -x_3 = -\lambda$

$$2x_1 = -4x_2 - 6x_3 = 4\lambda - 6\lambda = -2\lambda$$
$$x_1 = -\lambda$$

So null space is $\left\{ \begin{bmatrix} -\lambda \\ -\lambda \\ \lambda \end{bmatrix} \right\} = \left\{ \lambda \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

3. Notice that $\underline{v}_1 - \underline{v}_2 = \underline{v}_3$. Also $\underline{v}_1, \underline{v}_2$ are linearly independent because they are not multiples of each other. So $\mathcal{B} = \{\underline{v}_1, \underline{v}_2\}$ is a basis for the space. The dimension is then 2.

4. $$\begin{bmatrix} 0 & -2 & 4 \\ 1 & 1 & 1 \\ 6 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 6 & 0 & 0 \\ 0 & -2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -6 & -6 \\ 0 & -2 & 4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 6 \end{bmatrix}$$

There are three pivot columns. So

$\begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ are a basis for the column space.

5. $\left[\begin{array}{ccc|c} 2 & 3 & 0 & 2 \\ 0 & 0 & 4 & 1 \\ 3 & 2 & 0 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 3 & 0 & 2 \\ 3 & 2 & 0 & 2 \\ 0 & 0 & 4 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 3 & 0 & 2 \\ 0 & -5/2 & 0 & -1 \\ 0 & 0 & 4 & 1 \end{array}\right]$

$c_3 = \frac{1}{4}$

$-\frac{5}{2} c_2 = -1 \Rightarrow c_2 = \frac{2}{5}$

$2c_1 = -3c_2 + 2 = -\frac{6}{5} + \frac{10}{5} = \frac{4}{5} \Rightarrow c_1 = \frac{2}{5}$

So coordinates w.r.t. $\mathcal{B}$ are $\left(\frac{2}{5}, \frac{2}{5}, \frac{1}{4}\right)$.

6. $\det \begin{pmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 16 = 4 - 4\lambda + \lambda^2 - 16$

$= \lambda^2 - 4\lambda - 12 = (\lambda - 6)(\lambda + 2)$

So $\lambda = 6, -2$.

$\lambda = -2$: $\left[\begin{array}{cc|c} 4 & 4 & 0 \\ 4 & 4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 4 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_2 = \lambda$

$x_1 = -x_2 = -\lambda$ eigenspace is $\begin{bmatrix} -\lambda \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = 6$: $\left[\begin{array}{cc|c} -4 & 4 & 0 \\ 4 & -4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -4 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_2 = \lambda$

$x_1 = x_2 = \lambda$ eigenspace is $\begin{bmatrix} \lambda \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

7. $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ so $P^{-1} = \frac{1}{-2} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$D = P^{-1}AP = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 6 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 6 \end{bmatrix}$.

8. $\det \begin{bmatrix} 4-\lambda & 0 & 2 \\ -2 & 2-\lambda & 0 \\ 3 & 0 & 1-\lambda \end{bmatrix}$

$= (4-\lambda)\left[(2-\lambda)(1-\lambda) - 0\right] + 2\left[(-2)\cdot 0 - 3(2-\lambda)\right]$

$= (4-\lambda)(2-\lambda)(1-\lambda) - 4 - 6(2-\lambda)$

$= (4-\lambda)(2-3\lambda+\lambda^2) - 4 - 12 + 6\lambda$

$= 8 - 12\lambda + 4\lambda^2 - 2\lambda + 3\lambda^2 - \lambda^3 - 16 + 6\lambda$

$= -\lambda^3 + 7\lambda^2 - 8\lambda - 8$

9. $\cos\theta = \dfrac{\langle 2,1,3\rangle \cdot \langle 1,2,4\rangle}{\|\langle 2,1,3\rangle\| \cdot \|\langle 1,2,4\rangle\|}$

$= \dfrac{2+2+12}{\sqrt{4+1+9}\cdot\sqrt{1+4+16}} = \dfrac{16}{\sqrt{14}\sqrt{21}}$

10. $\text{proj}_{\underline{u}}\, \underline{y} = \dfrac{\underline{y}\cdot\underline{u}}{\underline{u}\cdot\underline{u}}\underline{u} = \dfrac{\langle 3,1,2\rangle\cdot\langle 1,1,3\rangle}{\langle 1,1,3\rangle\cdot\langle 1,1,3\rangle}\langle 1,1,3\rangle$

$= \dfrac{3+1+6}{1+1+9}\langle 1,1,3\rangle = \dfrac{10}{11}\langle 1,1,3\rangle$

$= \left\langle \dfrac{10}{11}, \dfrac{10}{11}, \dfrac{30}{11}\right\rangle.$

11. $\text{proj}_W \langle 2,1,2\rangle = \dfrac{\langle 2,1,2\rangle\cdot\langle 3,0,2\rangle}{\langle 3,0,2\rangle\cdot\langle 3,0,2\rangle}\langle 3,0,2\rangle$

$+ \dfrac{\langle 2,1,2\rangle\cdot\langle 2,0,3\rangle}{\langle 2,0,3\rangle\cdot\langle 2,0,3\rangle}\langle 2,0,3\rangle$

$= \dfrac{4+0+6}{9+0+4}\langle 3,0,2\rangle + \dfrac{4+0+6}{4+0+9}\langle 2,0,3\rangle$

$= \dfrac{10}{13}\langle 3,0,2\rangle + \dfrac{10}{13}\langle 2,0,3\rangle = \left\langle \dfrac{50}{13}, 0, \dfrac{50}{13}\right\rangle.$

$$\| \langle 2,1,2\rangle - \langle \tfrac{50}{13}, 0, \tfrac{50}{13}\rangle \| = \sqrt{\left(-\tfrac{24}{13}\right)^2 + (1)^2 + \left(-\tfrac{24}{13}\right)^2}$$

$$= \sqrt{\frac{496}{169} + \frac{169}{169} + \frac{496}{169}} = \sqrt{\frac{1161}{169}} = \frac{\sqrt{1161}}{13}$$

12. $\langle 2,1,6\rangle \cdot \langle -3,0,1\rangle = -6+0+6 = 0$ so these are $\perp$

$\langle 2,1,6\rangle \cdot \langle 1,0,3\rangle = 2+0+18 = 20 \neq 0$ so not $\perp$

$\langle -3,0,1\rangle \cdot \langle 1,0,3\rangle = -3+0+3 = 0$ so these are $\perp$.

13. Notice that $\langle 2,1,3\rangle - \langle 2,4,1\rangle = \langle 0,-3,2\rangle$.
Also $\langle 2,1,3\rangle$ and $\langle 2,4,1\rangle$ are linearly independent because they are not multiples of each other. So

$$\beta = \{\langle 2,1,3\rangle, \langle 2,4,1\rangle\}$$

is a basis for the space.

14. $\underline{v}_1 = \langle 1,4,2\rangle$

$$v_2 = \langle 6,3,2\rangle - \frac{\langle 6,3,2\rangle \cdot \langle 1,4,2\rangle}{\langle 1,4,2\rangle \cdot \langle 1,4,2\rangle} \langle 1,4,2\rangle$$

$$= \langle 6,3,2\rangle - \frac{6+12+4}{1+16+4} \langle 1,4,2\rangle$$

$$= \langle 6,3,2\rangle - \frac{22}{21} \langle 1,4,2\rangle$$

$$= \langle 6,3,2\rangle - \langle \tfrac{22}{21}, \tfrac{88}{21}, \tfrac{44}{21}\rangle$$

$$= \langle \tfrac{126}{21}, \tfrac{63}{21}, \tfrac{42}{21}\rangle - \langle \tfrac{22}{21}, \tfrac{88}{21}, \tfrac{44}{21}\rangle$$

15. The vector $\langle -4, 0, 1\rangle$ is $\perp$ to $\langle 1, 3, 4\rangle$ because $\langle -4, 0, 1\rangle \cdot \langle 1, 3, 4\rangle = -4 + 0 + 4 = 0$.

A unit vector in the same direction is

$$\left\langle \frac{-4}{\sqrt{17}}, 0, \frac{1}{\sqrt{17}} \right\rangle.$$