MATH 309

SOLUTIONS TO HOMEWORK 8

Section 4, 2

2. $\begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

So $\begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$ is in the null space of A.

4. $\left[\begin{array}{cccc|c} 1 & -6 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{array}\right]$

x_4 is free $x_4 = \lambda$

$x_3 = 0$

x_2 is free $x_2 = \mu$

$x_1 = 6x_2 - 4x_3 = 6\mu$

So the elements of the null space are given by

$$\begin{bmatrix} 6\mu \\ \mu \\ 0 \\ \lambda \end{bmatrix} = \mu \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The vectors $\begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ span the null space.

9. $\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{array}{l} a - 2b = 4c \\ 2a = c + 3d \end{array} \right\} = V$

If $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ and $\begin{bmatrix} a' \\ b' \\ c' \\ d' \end{bmatrix}$ are in V then

$$a - 2b = 4c \qquad a' - 2b' = 4c'$$
$$2a = c + 3d \qquad 2a' = c' + 3d'$$

$$(a + a') - 2(b + b') = 4(c + c')$$
$$2(a + a') = (c + c') + 3(d + d')$$

so $\begin{bmatrix} a + a' \\ b + b' \\ c + c' \\ d + d' \end{bmatrix}$ is in V. Now let $\lambda \in \mathbb{R}$.

Also $\begin{array}{l} \lambda a - 2\lambda b = 4\lambda c \\ 2\lambda a = \lambda c + 3\lambda d \end{array}$ so $\begin{bmatrix} \lambda a \\ \lambda b \\ \lambda c \\ \lambda d \end{bmatrix}$ is in V.

Finally, $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is in V.

So V is a vector space.

10. $\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} ; \begin{array}{l} a + 3b = c \\ b + c + a = d \end{array} \right\} = W$

If $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ and $\begin{bmatrix} a' \\ b' \\ c' \\ d' \end{bmatrix}$ are in $\bar{W}$ then

$$a + 3b = c \qquad a' + 3b' = c'$$
$$b + c + a = d \qquad b' + c' + a' = d'$$

So $(a+a') + 3(b+b') = c + c'$
$(b+b') + (c+c') + (a+a') = d + d'$

So $\begin{bmatrix} a+a' \\ b+b' \\ c+c' \\ d+d' \end{bmatrix}$ is in $\bar{W}$.

Also, if $\lambda \in \mathbb{R}$, then

$$\lambda a + 3\lambda b = \lambda c$$
$$\lambda b + \lambda c + \lambda a = \lambda d$$

so $\begin{bmatrix} \lambda a \\ \lambda b \\ \lambda c \\ \lambda d \end{bmatrix} \in \bar{W}$.

Finally, $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \bar{W}$.

So $\bar{W}$ is a vector space.

Section 4.3

2. These vectors do not span $\mathbb{R}^3$ because

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in the span.

$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are linearly independent but

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not.

4. $$\begin{bmatrix} 2 & 1 & -7 \\ -2 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & -7 \\ 0 & -2 & -2 \\ 0 & 3/2 & 15/2 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & -7 \\ 0 & -2 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

There are three pivot columns. So

$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \\ 4 \end{bmatrix}$ are linearly independent.

Hence they span $\mathbb{R}^3$.

9. $$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -5 & 4 & 0 \\ 3 & -2 & 1 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -5 & 4 & 0 \\ 0 & -2 & 10 & -8 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -5 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_4 is free so $x_4 = \lambda$

x_3 is free so $x_3 = \mu$

$x_2 = 5x_3 - 4x_4 = 5\mu - 4\lambda$

$x_1 = 3x_3 - 2x_4 = 3\mu - 2\lambda$

Solutions are

$$\begin{bmatrix} 3\mu-2\lambda \\ 5\mu-4\lambda \\ \mu \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix}$$

A basis for the null space is

$$\begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix}.$$

15. $$\begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ 0 & 2 & -8 & -5 & 0 \\ 0 & -3 & 12 & 5 & 5 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 4 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & 3 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Colums 1, 2, 4 are pivot columns. So

$$\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix}$$

are a basis for the space spanned by the given vectors.

Section 4.4

2. $\underline{x} = 8\begin{bmatrix}4\\5\end{bmatrix} - 5\begin{bmatrix}6\\7\end{bmatrix} = \begin{bmatrix}2\\5\end{bmatrix}$

6. $\left[\begin{array}{cc|c}1 & 5 & 4\\ -2 & -6 & 0\end{array}\right] \to \left[\begin{array}{cc|c}1 & 5 & 4\\ 0 & 4 & 8\end{array}\right]$

$$c_2 = 2$$
$$c_1 = -5c_2 + 4 = -6$$

So $\begin{bmatrix}4\\0\end{bmatrix} = -6\begin{bmatrix}1\\-2\end{bmatrix} + 2\begin{bmatrix}5\\-6\end{bmatrix}$

10. The change of coordinates matrix is

$$\begin{bmatrix}3 & 2 & 8\\ -1 & 0 & -2\\ 4 & -5 & 7\end{bmatrix}$$

12. The chang of coords matrix is

$$A = \begin{bmatrix}4 & 6\\ 5 & 7\end{bmatrix}$$

Its inverse is given by

$$\left[\begin{array}{cc|cc}4 & 6 & 1 & 0\\ 5 & 7 & 0 & 1\end{array}\right] \to \left[\begin{array}{cc|cc}4 & 6 & 1 & 0\\ 0 & -\frac{1}{2} & -\frac{5}{4} & 1\end{array}\right]$$

$$\to \left[\begin{array}{cc|cc}4 & 0 & -14 & +12\\ 0 & -\frac{1}{2} & -5/4 & 1\end{array}\right] \to \left[\begin{array}{cc|cc}1 & 0 & -\frac{7}{2} & +3\\ 0 & 1 & 5/2 & -2\end{array}\right]$$

So $A^{-1} = \begin{bmatrix}-7/2 & +3\\ 5/2 & -2\end{bmatrix}$

$$A^{-1}\underline{x} = \begin{bmatrix} -7/2 & 3 \\ 5/2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$$

So $[\underline{x}]_{\mathcal{B}} = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$

Section 4.5

2. $\left\{ \begin{bmatrix} 4s \\ -3s \\ -t \end{bmatrix} ; s, t \in \mathbb{R} \right\}$

So a typical element is

$$s\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

A basis for the space is $\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$.

The dimension of the space is 2.

6. $\left\{ \begin{bmatrix} 3a+6b-c \\ 6a-2b-2c \\ -9a+5b+3c \\ -3a+b+c \end{bmatrix} ; a, b, c \in \mathbb{R} \right\}$

A typical element is

$$a\begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix} + b\begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix} + c\begin{bmatrix} -1 \\ -2 \\ 3 \\ 1 \end{bmatrix}.$$

We need to determine whether these three vectors are linearly independent

$$\begin{bmatrix} 3 & 6 & -1 \\ 6 & -2 & -2 \\ -9 & 5 & 3 \\ -3 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 6 & -1 \\ 0 & -14 & 0 \\ 0 & 23 & 0 \\ 0 & 7 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 6 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 7 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 3 & 6 & -1 \\ 0 & 7 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

There are two pivot colums.

So $\begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix}$ form a basis.

The dimension of the space is 2.

10. $$\begin{bmatrix} 1 & -3 & -8 & -3 \\ -2 & 4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & -2 & -10 & -6 \\ 0 & 1 & 5 & 7 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & 0 & 0 & 8 \\ 0 & 1 & 5 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

Columns 1, 2, 4 are pivot columns.

So $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix}$ are a basis,

The dimension is 3.

12. $$A = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

columns 1, 3, 4 are pivot columns.

The dimension of the column space is 3.

Rows 1, 2, 3 are pivot rows. The dimension of the row space is 3.

$$\left[\begin{array}{cccccc|c} 1 & 3 & -4 & 2 & -1 & 6 & 0 \\ 0 & 0 & 1 & -3 & 7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_6 is free $x_6 = \lambda$

x_5 is free $x_5 = \mu$

x_2 is free $x_2 = \tau$

$x_4 = -4x_5 + 3x_6 = -4\mu + 3\lambda$

$x_3 = 3x_4 - 7x_5 = 3(-4\mu + 3\lambda) - 7\mu$

$= -19\mu + 9\lambda$

$x_1 = -3x_2 + 4x_3 - 2x_4 + x_5 - 6x_6$

$= -3\tau + 4(-19\mu + 9\lambda) - 2(-4\mu + 3\lambda)$

$+ \mu - 6\lambda$

$= -3\tau - 76\mu + 36\lambda + 8\mu - 6\lambda + \mu - 6\lambda$

$= -3\tau - 67\mu + 24\lambda$

So a typical element of the null space is

$$\begin{bmatrix} -3\tau - 67\mu + 24\lambda \\ \tau \\ -19\mu + 9\lambda \\ -4\mu + 3\lambda \\ \mu \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} 24 \\ 0 \\ 9 \\ 3 \\ 0 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -67 \\ 0 \\ -19 \\ -4 \\ 1 \\ 0 \end{bmatrix} + \tau \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The null space is three dimensional.

Section 4.6

2a) Let $\underline{c}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $\underline{c}_2 = \begin{bmatrix}0\\1\end{bmatrix}$.

Then $\underline{b}_1 = -\underline{c}_1 + 4\underline{c}_2 = \begin{bmatrix}-1\\0\end{bmatrix} + \begin{bmatrix}0\\4\end{bmatrix} = \begin{bmatrix}-1\\4\end{bmatrix}$

$\underline{b}_2 = 5\underline{c}_1 - 3\underline{c}_2 = \begin{bmatrix}5\\0\end{bmatrix} + \begin{bmatrix}0\\-3\end{bmatrix} = \begin{bmatrix}5\\-3\end{bmatrix}$

$$\left[\begin{array}{cc|cc}\underline{b}_1 & \underline{b}_2 & \underline{c}_1 & \underline{c}_2\end{array}\right] = \left[\begin{array}{cc|cc}-1 & 5 & 1 & 0\\ 4 & -3 & 0 & 1\end{array}\right].$$

So the change of coordinates basis from $\mathcal{C}$ to $\mathcal{B}$ is

$$\begin{bmatrix}-1 & 5\\ 4 & -3\end{bmatrix}.$$

b) $\underline{x} = 5\underline{b}_1 + 3\underline{b}_2 = 5\begin{bmatrix}-1\\4\end{bmatrix} + 3\begin{bmatrix}5\\-3\end{bmatrix} = \begin{bmatrix}10\\11\end{bmatrix}$.

8) $\left[\begin{array}{cc|cc}\underline{b}_1 & \underline{b}_2 & \underline{c}_1 & \underline{c}_2\end{array}\right] = \left[\begin{array}{cc|cc}-1 & 1 & 1 & 1\\ 8 & -5 & 4 & 1\end{array}\right] \to \left[\begin{array}{cc|cc}-1 & 1 & 1 & 1\\ 0 & 3 & 12 & 9\end{array}\right]$

$\to \left[\begin{array}{cc|cc}-1 & 0 & -3 & -2\\ 0 & 3 & 12 & 9\end{array}\right] \to \left[\begin{array}{cc|cc}1 & 0 & 3 & 2\\ 0 & 1 & 4 & 3\end{array}\right]$

So the change of coord matrix from $\mathcal{C}$ to $\mathcal{B}$ is

$$\begin{bmatrix}3 & 2\\ 4 & 3\end{bmatrix}$$

$\left[\begin{array}{cc|cc}3 & 2 & 1 & 0\\ 4 & 3 & 0 & 1\end{array}\right] \to \left[\begin{array}{cc|cc}3 & 2 & 1 & 0\\ 0 & 1/3 & -\frac{4}{3} & 1\end{array}\right] \to \left[\begin{array}{cc|cc}3 & 0 & 9 & -6\\ 0 & 1/3 & -4/3 & 1\end{array}\right]$

$\to \left[\begin{array}{cc|cc}1 & 0 & 3 & -2\\ 0 & 1 & -4 & 3\end{array}\right]$

So the change of coords. matrix from $\mathcal{B}$ to $\mathcal{C}$ is $\begin{bmatrix}3 & -2\\ -4 & 3\end{bmatrix}$

10. $\left[\underline{b}_1\ \underline{b}_2 \middle| \underline{c}_1\ \underline{c}_2\right] = \left[\begin{array}{cc|cc} 7 & 2 & 4 & 5 \\ -2 & -1 & 1 & 2 \end{array}\right]$

$$\to \left[\begin{array}{cc|cc} 7 & 2 & 4 & 5 \\ 0 & -\frac{3}{7} & \frac{15}{7} & \frac{24}{7} \end{array}\right] \to \left[\begin{array}{cc|cc} 7 & 0 & \frac{294}{21} & \frac{351}{21} \\ 0 & -\frac{3}{7} & \frac{15}{7} & \frac{24}{7} \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & 0 & \frac{294}{147} & \frac{351}{147} \\ 0 & 1 & -5 & -8 \end{array}\right]$$

The change of coords matrix from $\mathcal{C}$ to $\mathcal{B}$ is

$$\begin{bmatrix} \frac{294}{147} & \frac{351}{147} \\ -5 & -8 \end{bmatrix}$$

The change of coords matrix from $\mathcal{B}$ to $\mathcal{C}$ is the inverse of this matrix.