Math 309
Krantz

Spring, 2023

FIRST MIDTERM EXAM

General Instructions: Read the statement of each problem carefully. If you want full credit on a problem then you must show your work. If you only write the answer then you will *not* receive full credit.

Be sure to ask questions if anything is unclear. This exam has 8 questions and is worth 100 points. You will have 50 minutes to take this exam.

(12 points) **1.** Use an augmented matrix to solve this system of linear equations.

$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= -4 \\ 3x_1 - 7x_2 + 7x_3 &= -8 \\ -4x_1 + 6x_2 + 2x_3 &= 4 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 3 & -7 & 7 & -8 \\ -4 & 6 & 2 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 18 & -12 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

So $3x_3 = 0 \Rightarrow x_3 = 0$

$2x_2 = 5x_3 + 4 = 4 \Rightarrow x_2 = 2$

$1x_1 = 3x_2 - 4x_3 - 4 = 6 + 0 - 4 = 2$

$\{x_1, x_2, x_3\} = \{2, 2, 0\}$.

(12 points) **2.** Find the general solution of the system whose augmented matrix is give here.

$$\left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ -1 & 7 & -4 & 2 & 7 \\ 0 & 0 & 1 & -2 & -3 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & -4 & 8 & 12 \\ 0 & 0 & 1 & -2 & -3 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & -4 & 8 & 12 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_4 is free so $x_4 = \lambda$, x_2 is free so $x_2 = \mu$.

$-4x_3 = -8x_4 + 12 = -8\lambda + 12 \Rightarrow x_3 = 2\lambda - 3$

$x_1 = 7x_2 - 6x_4 + 5 = 7\mu - 6\lambda + 5$

So solution set is

$\{x_1, x_2, x_3, x_4\} = \{7\mu - 6\lambda + 5, \mu, 2\lambda - 3, \lambda\}$.

(12 points) **3.** Determine whether $\mathbf{b}$ is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 is free so $x_3 = \lambda$

$x_2 = -4x_3 + 3 = -4\lambda + 3$

$x_1 = -5x_3 + 2 = -5\lambda + 2$

Take $\lambda = 1$ so

$x_1 = -3$, $x_2 = -1$, $x_3 = 1$.

Then

$\underline{b} = -3\underline{v}_1 - 1\underline{v}_2 + 1\underline{v}_3$

(12 points) **4.** Given A and $\mathbf{b}$, write the augmented matrix for the linear system that corresponds to the matrix equation $A\mathbf{x} = \mathbf{b}$. Then solve the system and write the solution as a vector.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -3 & -1 & 2 & 1 \\ 0 & 5 & 3 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 5 & 3 & -1 \end{array}\right].$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 0 & -2 & -2 \end{array}\right]$$

So $x_3 = 1$

$5x_2 = -5x_3 + 1 = -4 \Rightarrow x_2 = \frac{-4}{5}$

$1x_1 = -2x_2 - x_3 = \frac{8}{5} - 1 = \frac{3}{5}$

Solution is $\begin{bmatrix} 3/5 \\ -4/5 \\ 1 \end{bmatrix}$.

(14 points) **5.** Write the solution of the given homogeneous system in paremetric vector form (using λ, μ, etc.).

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 0 \\ x_1 + 4x_2 - 8x_3 &= 0 \\ -3x_1 - 7x_2 + 9x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 0 \\ 1 & 4 & -8 & 0 \\ -3 & -7 & 9 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 is free so $x_3 = \lambda$

$x_2 = 3x_3 = 3\lambda$

$x_1 = -3x_2 + 5x_3 = -9\lambda + 5\lambda = -4\lambda$

So solution is $\{-4\lambda, 3\lambda, \lambda\}$.

(14 points) **6.** Determine whether the columns of this matrix form a linearly independent set.

$$\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -4 & -3 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 1 & 0 & 3 & 0 \\ 5 & 4 & 6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ -4 & -3 & 0 & 0 \\ 5 & 4 & 6 & 0 \\ 0 & -1 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & -3 & 12 & 0 \\ 0 & 4 & -9 & 0 \\ 0 & -1 & 4 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & -3 & 12 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = 0$

$-3x_2 = -12x_3 = 0$ so $x_2 = 0$

$x_1 = -3x_3 = 0$

So there are only trivial solutions to this system.

Hence the columns are linearly independent.

(12 points) **7.** Given the matrix A, define the linear mapping T by $T\mathbf{x} = A\mathbf{x}$. Find a vector $\mathbf{x}$ whose image under T is $\mathbf{b}$. Say whether or not $\mathbf{x}$ is unique.

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 3 & -5 & -9 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 6 \\ -7 \\ -9 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 3 & -5 & -9 & -9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 4 & -15 & -27 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$x_3 = 1$

$x_2 = 4x_3 - 7 = -3$

$x_1 = 3x_2 - 2x_3 + 6 = -9 - 2 + 6 = -5$

Solution is $\{-5, -3, 1\}$. It is unique.

(12 points) **8.** The map T from $\mathbf{R}^3$ to $\mathbf{R}^2$ satisfies $T(\mathbf{e}_1) = (1,3)$, $T(\mathbf{e}_2) = (4,2)$, $T(\mathbf{e}_3) = (-5,4)$. [Here $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the columns of the 3×3 identity matrix.] Find the standard matrix of T.

$$A = \begin{bmatrix} 1 & 4 & -5 \\ 3 & 2 & 4 \end{bmatrix}.$$