Math 309
Krantz

Spring, 2023
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SECOND MIDTERM EXAM

General Instructions: Read the statement of each problem carefully. If you want full credit on a problem then you must show your work. If you only write the answer then you will *not* receive full credit.

Be sure to ask questions if anything is unclear. This exam has 8 questions and is worth 100 points. You will have 50 minutes to take this exam.

(12 points) **1.** Calculate the inverse of the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

So the inverse is $\begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$

(12 points) **2.** Let

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}.$$

Define

$$\mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}.$$

Is $\mathbf{b}$ in the column space of A?

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ 0 & -2 & -6 & 5 \\ 0 & -6 & -18 & 15 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ 0 & -2 & -6 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 free so $x_3 = \lambda$

$-2x_2 = 6x_3 + 5 = 6\lambda + 5$

$x_2 = -3\lambda - 5/2$

$x_1 = 3x_2 + 4x_3 + 3$

$= 3(-3\lambda - 5/2) + 4\lambda + 3$

$= -5\lambda - \frac{9}{2}$. Take $\lambda = 1$

So $\begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix} = -\frac{19}{2}\begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix} - \frac{11}{2}\begin{bmatrix} -3 \\ 6 \\ 7 \end{bmatrix} + 1\begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}$

Yes, $\underline{\underline{b}}$ is in the column space of A.

(12 points) **3.** Find a basis for the null space of

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 free so $x_3 = \lambda$

$x_2 = 0$

$x_1 = -x_3 = -\lambda$

null space is $\begin{bmatrix} -\lambda \\ 0 \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$

Thus $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is a basis for the null space.

(12 points) **4.** Find a basis for the column space of

$$B = \begin{bmatrix} 0 & -3 & 5 \\ 1 & 2 & -1 \\ 9 & 0 & 0 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 0 & -3 & 5 & 0 \\ 1 & 2 & -1 & 0 \\ 9 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 5 & 0 \\ 9 & 0 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 5 & 0 \\ 0 & -18 & 9 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 5 & 0 \\ 0 & 0 & -21 & 0 \end{array}\right]$$

$x_3 = 0$

$x_2 = 0$

$x_1 = 0.$

So $\begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$

are linearly independent, So these three form a basis for the column space of B.

(14 points) **5.** Calculate the determinant of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 4 & 4 & 4 \end{bmatrix}.$$

$$\det A = 1 \cdot (2 \cdot 4 - 4 \cdot 1) - 2(3 \cdot 4 - 4 \cdot 1) + 3(3 \cdot 4 - 4 \cdot 2)$$

$$= (8-4) - 2(12-4) + 3(12-8)$$

$$= 4 - 16 + 12 = 0.$$

(14 points) **6.** Use Cramer's Rule to solve the system

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8 \end{aligned}$$

$$x_1 = \frac{\det \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}}{\det \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}} = \frac{24+16}{12-10} = \frac{40}{2} = 20$$

$$x_2 = \frac{\det \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}}{\det \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}} = \frac{24+30}{2} = \frac{54}{2} = 27$$

(12 points) **7.** Let

$$A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}.$$

Verify that

$$\det(AB) = [\det A] \cdot [\det B].$$

$$AB = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix}$$

$$\det[AB] = 325 - 280 = 45$$

$$\det A = 12 - 3 = 9$$

$$\det B = 8 - 3 = 5$$

$$[\det A] \cdot [\det B] = 9 \cdot 5 = 45 = \det(AB).$$

(12 points) **8.** Are the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 3 \\ -5 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix}$$

linearly independent? Why or why not?

$$\left[\begin{array}{ccc|c} 5 & -3 & 2 & 0 \\ -7 & 3 & -7 & 0 \\ 9 & -5 & 5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 5 & -3 & 2 & 0 \\ 0 & -6/5 & -\frac{21}{5} & 0 \\ 0 & 2/5 & 7/5 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 5 & -3 & 2 & 0 \\ 0 & -6/5 & -21/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 is free so $x_3 = \lambda$

$-6/5\, x_2 = \frac{21}{5} x_3 = \frac{21}{5}\lambda$

$x_2 = -\frac{105}{30}\lambda = -\frac{7}{2}\lambda$

$5x_1 = 3x_2 - 2x_3$

$= -\frac{21}{2}\lambda - 2\lambda = -\frac{25}{2}\lambda$

$x_1 = -\frac{5}{2}\lambda$

Take $\lambda = 1$. So

$$-\frac{5}{2}\begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix} - \frac{7}{2}\begin{bmatrix} -3 \\ 3 \\ -5 \end{bmatrix} + 1\begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix} = \underline{0}$$

The vectors are linearly <u>dependent</u>.