Math 310
September 14, 2020 Lecture

Steven G. Krantz

August 27, 2020

Figure: This is your instructor.

Principles of Logic

Strictly speaking, our approach to logic is "intuitive" or "naïve." Whereas in ordinary conversation these emotion-charged words may be used to downgrade the value of that which is being described, our use of these words is more technical. What is meant is that we shall prescribe in this chapter certain rules of logic which are to be followed in the rest of the book. They will be presented to you in such a way that their validity should be intuitively appealing and self-evident. We cannot *prove* these rules. The rules of logic are the point where our learning begins. A more advanced course in logic will explore other logical methods. The ones that we present here are universally accepted in mathematics and in most of science.

We shall begin with sentential logic and elementary connectives. This material is called the *propositional calculus* (to distinguish it from the predicate calculus, which will be treated later). In other words, we shall be discussing *propositions*—which are built up from atomic statements and connectives. The elementary connectives include "and," "or," "not," "if-then," and "if and only if." Each of these will have a precise meaning and will have exact relationships with the other connectives.

An *atomic statement* (or *elementary statement*) is a sentence with a subject and a verb (and sometimes an object) but no connectives (and, or, not, if-then, if-and-only-if). For example,

John is good.

Mary has bread.

Ethel reads books.

are all atomic statements. We build up sentences, or propositions, from atomic statements using connectives.

After we treat the connectives we shall consider the quantifiers “for all” and “there exists” and their relationships with the connectives from the last paragraph. The quantifiers will give rise to the so-called *predicate calculus*. Connectives and quantifiers will prove to be the building blocks of all future statements in this book, indeed in all of mathematics.

In everyday conversation, people sometimes argue about whether a statement is true or not. In mathematics there is nothing to argue about. In practice a sensible statement in mathematics is either true or false, and there is no room for opinion about this attribute. How do we determine which statements are true and which are false?

The modern methodology in mathematics works as follows:

- We *define* certain terms.
- We *assume* that these terms have certain properties or truth attributes (these assumptions are called axioms).
- We specify certain rules of logic.

Any statement that can be derived from the axioms, using the rules of logic, is understood to be true (we call such a derivation a *proof*). The statement that we have so derived is called a *theorem* or a *proposition* or perhaps a *lemma*. It is not necessarily the case that every true statement can be derived in this fashion. However, in practice this is our method for verifying that a statement is true.

On the other hand, a statement is false if it is inconsistent with the axioms and the rules of logic. That is to say, a statement is false if the assumption that it is true leads to a contradiction. Alternatively, a statement P is false if the negation of P can be established or proved. While it is possible for a statement to be false without our being able to derive a contradiction in this fashion, in practice we establish falsity by the method of contradiction or by giving a counterexample (which is another aspect of the method of contradiction).

The point of view being described here is special to mathematics. While it is indeed true that mathematics is used to model the world around us—in physics, engineering, and in other sciences—the subject of mathematics itself is a man-made system. Its internal coherence is guaranteed by the axiomatic method that we have just described.

It is reasonable to ask whether mathematical truth is a construct of the human mind or an immutable part of nature. For instance, is the assertion that "the area of a circle is π times the radius squared" actually a fact of nature just like Newton's inverse square law of gravitation? Our point of view is that mathematical truth is relative.

The formula for the area of a circle is a logical consequence of the axioms of mathematics, nothing more. The fact that the formula seems to describe what is going on in nature is convenient, and is part of what makes mathematics useful. But that aspect is something over which we as mathematicians have no control. Our concern is with the internal coherence of our logical system.

It can be asserted that a proof (a concept to be discussed and developed later in the book) is a psychological device for convincing the reader that an assertion is true. However, our view in this book is more rigid: a proof of an assertion is a sequence of applications of the rules of logic to derive the assertion from the axioms. There is no room for opinion here. The axioms are plain. The rules are rigid. A proof is like a sequence of moves in a game of chess. If the rules are followed, then the proof is correct; otherwise not.

"And" and "Or"

Let A and B be atomic statements such as "Chelsea is smart" or "The earth is flat." The statement

$$\text{“}A \quad \text{and} \quad B\text{”}$$

means that both A is true *and* B is true. For instance, Arvid is old and Arvid is fat.

means both that Arvid is old *and* Arvid is fat. If we meet Arvid and he turns out to be young and fat, then the statement is false. If he is old and thin then the statement is false. Finally, if Arvid is *both* young and thin then the statement is false. The statement is *true* precisely when both properties—oldness and fatness—hold. We may summarize these assertions with a *truth table*. We let

$$A = \text{ Arvid is old.}$$

and

$$B = \text{ Arvid is fat.}$$

The expression

$$A \wedge B$$

will denote the phrase "A and B." We call this statement the *conjunction* of A and B. The letters "T" and "F" denote "True" and "False," respectively. Then we have

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Notice that we have listed all possible truth values of A and B and the corresponding values of the *conjunction* $A \wedge B$. It is a good habit to learn to always write your truth tables in this fashion: with the `TT` line first, the `TF` line second, the `FT` line third, and the `FF` line last. That makes it easier to compare two truth tables.

In a restaurant, the menu often contains phrases such as soup or salad

This means that we may select soup *or* select salad, but we may not select both. This use of "or" is called the *exclusive* "or"; it is not the meaning of "or" that we use in mathematics and logic. In mathematics we instead say that "A or B" is true provided that A is true or B is true or *both* are true. This is the *inclusive* "or." If we let $A \vee B$ denote "A or B," then the truth table is

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

We call the statement $A \vee B$ the *disjunction* of A and B.

We see from the truth table that the only way that "A or B" can be false is if *both* A is false and B is false. For instance, the statement

Hillary is beautiful or Hillary is poor.

means that Hillary is either beautiful or poor or both. In particular, she will not be both ugly and rich. Another way of saying this is that if she is ugly she will compensate by being poor; if she is rich she will compensate by being beautiful. *But she could be both beautiful and poor.*

Example

The statement

$$x > 2 \quad \text{and} \quad x < 5$$

is true for the number $x = 3$ because this value of x is both greater than 2 *and* less than 5. It is false for $x = 6$ because this x value is greater than 2 but not less than 5. It is false for $x = 1$ because this x is less than 5 but not greater than 2.

Example

The statement

$$x \text{ is odd and } x \text{ is a perfect cube.}$$

is true for $x = 27$ because both assertions hold. It is false for $x = 7$ because this x, while odd, is not a cube. It is false for $x = 8$ because this x, while a cube, is not odd. It is false for $x = 10$ because this x is neither odd nor is it a cube.

Example

The statement

$$x < 3 \text{ or } x > 6$$

is true for $x = 2$ since this x is < 3 (even though it is not > 6). It holds (that is, it is true) for $x = 9$ because this x is > 6 (even though it is not < 3). The statement fails (that is, it is false) for $x = 4$ since this x is neither < 3 nor > 6.

Example

The statement

$$x > 1 \ \text{ or } \ x < 4$$

is true for every real x.

Example

The statement $(A \vee B) \wedge B$ has the following truth table:

A	B	$A \vee B$	$(A \vee B) \wedge B$
T	T	T	T
T	F	T	F
F	T	T	T
F	F	F	F

Notice in Example 1.3.5 that the statement $(A \vee B) \wedge B$ has the same truth values as the simpler statement B. In what follows, we shall call such pairs of statements (having the same truth values) *logically equivalent*.

The words “and” and “or” are called *connectives*: their role in sentential logic is to enable us to build up (or to connect together) pairs of atomic statements. The idea is to use very simple statements, like “ Jennifer is swift” as building blocks; then we compose more complex statements from these building blocks by using connectives.

You will notice that we *always* lay out the truth values for A and B in a truth table in the same way. This makes it easier for us to compare the truth tables of different statements.

In the next two sections, we will become acquainted with the other two basic connectives "not" and "if-then."