Math 310
September 16, 2020 Lecture

Steven G. Krantz

August 28, 2020

Figure: This is your instructor.

"Not"

The statement "not A," written $\sim A$, is true whenever A is false. For example, the statement

Charles is not happily married.

is true provided the statement "Charles is happily married" is false. The truth table for $\sim A$ is as follows:

A	$\sim A$
T	F
F	T

Greater understanding is obtained by combining connectives:

Example

Here is the truth table for $\sim(A \wedge B)$:

A	B	$A \wedge B$	$\sim(A \wedge B)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

Example

Now we look at the truth table for $(\sim A) \vee (\sim B)$:

A	B	$\sim A$	$\sim B$	$(\sim A) \vee (\sim B)$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Notice that the statements $\sim(A \wedge B)$ and $(\sim A) \vee (\sim B)$ have the *same truth table*. As previously noted, such pairs of statements are called *logically equivalent*.

The logical equivalence of $\sim(A \wedge B)$ with $(\sim A) \vee (\sim B)$ makes good intuitive sense: the statement $A \wedge B$ fails precisely when either A is false *or* B is false. Since in mathematics we cannot rely on our intuition to establish facts, it is important to have the truth table technique for establishing logical equivalence. The exercise set will give you further practice with this notion.

One of the main reasons that we use the *inclusive* definition of "or" rather than the exclusive one is so that the connectives "and" and "or" have the nice relationship just discussed. It is also the case that $\sim (A \vee B)$ and $(\sim A) \wedge (\sim B)$ are logically equivalent. These logical equivalences are sometimes referred to as *de Morgan's Laws*.

"If-Then"

A statement of the form "If A then B" asserts that, whenever A is true, then B is also true. This assertion (or "promise") is tested when A is true because it is then claimed that something else (namely B) is true as well. *However*, when A is false, then the statement "If A then B" *claims nothing*. Using the symbols $A \Rightarrow B$ to denote "If A then B," we obtain the following truth table:

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

Notice that we use here an important principle of Aristotelian logic: every sensible statement is either true or false. There is no "in between" status. When A is false, we can hardly assert that

$$A \Rightarrow B$$

is false. For $A \Rightarrow B$ asserts that "whenever A is true then B is true," and A is not true!
Put in other words, when A is false, then the statement $A \Rightarrow B$ is not tested. It therefore cannot be false. So it must be true.

Example

The statement "If $2 = 4$, then Calvin Coolidge was our greatest president" is true (the antecedent is false and the conclusion may be true or false). This is the case no matter what you think of Calvin Coolidge.

The statement "If fish have hair, then chickens have lips" is true (the antecedent is false, and the conclusion is false).

The statement "If $9 > 5$, then dogs don't fly" is true (the antecedent is true, and the conclusion is true).
[Notice that the "if" part of the sentence and the "then" part of the sentence need not be related in any intuitive sense. The truth or falsity of an "if-then" statement is simply a fact about the logical values of its hypothesis and of its conclusion.]

Example

The statement $A \Rightarrow B$ is logically equivalent to $(\sim A) \vee B$. The truth table for the latter is

A	B	$\sim A$	$(\sim A) \vee B$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

which is the same as the truth table for $A \Rightarrow B$.

You should think for a bit to see that $(\sim A) \vee B$ *says the same thing* as $A \Rightarrow B$. To wit, assume that the statement $(\sim A) \vee B$ is true. Now suppose that A is true. Then the first half of the disjunction is false; so the second half must be true. In other words, B must be true. But that says that $A \Rightarrow B$. For the converse, assume that $A \Rightarrow B$ is true. This means that if A holds, then B must follow. But this may be rephrased as saying that if the first half of the disjunction $(\sim A) \vee B$ is false, then the second half is true. That merely affirms the disjunction. So the two statements are equivalent, i.e., they say the same thing.

Once you believe that assertion, then the truth table for $(\sim A) \vee B$ gives us another way to understand the truth table for $A \Rightarrow B$.

kThere are in fact infinitely many pairs of logically equivalent statements. But just a few of these equivalences are really important in practice—most others are built up from these few basic ones. Some of the other basic pairs of logically equivalent statements are explored in the exercises.

Example

The statement

$$\text{If} \quad \text{x} \;\; \text{is negative, then} \;\; -5 \cdot x \;\; \text{is positive.}$$

is true. For if $x < 0$, then $-5 \cdot x$ is indeed > 0; if $x \geq 0$, then the statement is unchallenged.

Example

The statement

$$\text{If} \;\; (x > 0 \;\; \text{and} \;\; x^2 < 0), \quad \text{then} \;\; x \geq 10.$$

is true since the hypothesis "$(x > 0 \;\; \text{and} \;\; x^2 < 0)$" is never true.

Example

The statement

$$\text{If } \; x > 0, \quad \text{then } (x^2 < 0 \;\text{ or }\; 2x < 0).$$

is false since the conclusion "$(x^2 < 0 \;\text{ or }\; 2x < 0)$" is false whenever the hypothesis $x > 0$ is true.

Example

Let us construct a truth table for the statement
$(A \vee (\sim B)) \Rightarrow ((\sim A) \wedge B)$.

A	B	$\sim A$	$\sim B$	$(A \vee (\sim B))$	$((\sim A) \wedge B)$	$(A \vee (\sim B)) \Rightarrow ((\sim A) \wedge B)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	F	T	T	T	F	F

Notice that the statement $\big(A \vee (\sim B)\big) \Rightarrow \big((\sim A) \wedge B\big)$ has the same truth table as $\sim \big(B \Rightarrow A\big)$. Can you comment on the logical equivalence of these two statements?

Perhaps the most commonly used logical syllogism is the following. Suppose that we know the truth of A and of $A \Rightarrow B$. We wish to conclude B. Examine the truth table for $A \Rightarrow B$. The only line in which both A is true and $A \Rightarrow B$ is true is the line in which B is true. That justifies our reasoning. In logic texts, the syllogism we are discussing is known as *modus ponendo ponens*.

In fact, *modus ponendo ponens* is quite classical terminology, going back at least to the nineteenth century. It is not commonly used today.

Contrapositive, Converse, and "Iff"

The statement

If A then B

is the same as

$$A \Rightarrow B$$

or

A suffices for B

or as saying

A only if B

All these forms are encountered in practice, and you should think about them long enough to realize that they all say the same thing.

On the other hand,

$$\text{If } B \ \text{ then } A$$

is the same as saying

$$B \Rightarrow A$$

or

A is necessary for B

or as saying

$$A \ \text{ if } \ B$$

We call the statement $B \Rightarrow A$ the *converse* of $A \Rightarrow B$. The converse of a statement is logically distinct from that original statement; the truth or falsity of one is independent of the truth or falsity of the other. Our examples will illustrate this point.

Example

The converse of the statement

If x is a healthy horse, then x has four legs.

is the statement

If x has four legs, then x is a healthy horse.

Notice that these statements have very different meanings: the first statement is true, while the second (its converse) is false. For instance, a chair has four legs, but it is not a healthy horse.

Example

The converse of the statement

$$\text{If } x > 0, \text{ then } 2x > 0.$$

is the statement

$$\text{If } 2x > 0, \text{ then } x > 0.$$

Notice that both statements are true.

Example

The converse of the statement

$$\text{If } x > 0, \text{ then } x^2 > 0.$$

is the statement

$$\text{If } x^2 > 0, \text{ then } x > 0.$$

Notice that the first implication is true, while the second is false.

The statement

$$A \ \text{ if and only if } B$$

is a brief way of saying

$$\text{If } A \ \text{ then } \ B. \qquad \textit{and} \qquad \text{If } B \ \text{ then } \ A.$$

We abbreviate A if and only if B as $A \Leftrightarrow B$ or as A iff B. Here is a truth table for $A \Leftrightarrow B$:

A	B	$A \Rightarrow B$	$B \Rightarrow A$	$A \Leftrightarrow B$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T