Math 310
September 23, 2020 Lecture

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Figure: This is your instructor.

When a chemist asserts that a substance that is subjected to heat will tend to expand, he or she verifies the assertion through experiment. It is a consequence of the *definition* of heat that heat will excite the atomic particles in the substance; it is plausible that this in turn will necessitate expansion of the substance. However, our knowledge of nature is not such that we may turn these theoretical ingredients into a categorical proof. Additional complications arise from the fact that the word "expand" requires detailed definition. Apply heat to water that is at temperature $40°$ F or above, and it expands—with enough heat it becomes a gas that surely fills more volume than the original water. But apply heat to a diamond, and there is no apparent "expansion"—at least not to the naked eye.

Mathematics is a less ambitious subject. In particular, it is closed. It does not reach outside itself for verification of its assertions. When we make an assertion in mathematics, we must verify it using the rules that we have laid down. That is, we verify it by applying our rules of logic to our axioms and our definitions; in other words, we construct a *proof*. An earlier lecture contains some discussion of proofs and the rules of logic.

In modern mathematics we have discovered that there are perfectly sensible mathematical statements that in fact *cannot* be verified in this fashion, nor can they be proven false. This is a manifestation of Gödel's Incompleteness theorem: that any sufficiently complex logical system will contain such unverifiable, indeed untestable, statements. Fortunately, in practice, such statements are the exception rather than the rule. In this book, and in almost all of university-level mathematics, we concentrate on learning about statements whose truth or falsity *is* accessible by way of proof.

This chapter considers the notion of mathematical proof. We shall concentrate on the three principal types of proof: direct proof, proof by contradiction, and proof by mathematical induction. Some other miscellaneous methods of proof will be treated as well. This includes proof by enumeration, proof by exhaustion, proof by cases, proof by contraposition, and several others.

In practice, a mathematical proof may contain elements of several or all of these techniques. You will see all the basic elements here. You should be sure to master each of these proof techniques, both so that you can recognize them in your reading and so that they become tools that you can use in your own work.

In this lecture, we shall assume that you are familiar with the positive integers, or *natural numbers* (a detailed treatment of the natural numbers appears later). This number system $\{1, 2, 3, \ldots\}$ is denoted by the symbol $\mathbb{N}$. For now we will take the elementary arithmetic properties of $\mathbb{N}$ for granted. We shall formulate various statements about natural numbers and prove them. Our methodology will emulate the discussions in earlier lectures. We begin with a definition.

Definition

A natural number n is said to be *even* if, when it is divided by 2, there is no remainder.

Definition

A natural number n is said to be *odd* if, when it is divided by 2, the remainder is 1.

You may have never before considered, at this level of precision, what is the meaning of the terms "odd" or "even." But your intuition should confirm these definitions. A good definition should be precise, but it should also appeal to your heuristic idea about the concept that is being defined.

Notice that, according to these definitions, any natural number is either even or odd. For if n is any natural number, and if we divide it by 2, then the remainder will be either 0 or 1—there is no other possibility (according to the Division Algorithm—see [HER]). In the first instance, n is even; in the second, n is odd.

In what follows we will find it convenient to think of an even natural number as one having the form $2m$ for some natural number m. We will think of an odd natural number as one having the form $2k + 1$ (or sometimes $2k - 1$) for some natural number k. Check for yourself that, in the first instance, division by 2 will result in a quotient of m and a remainder of 0; in the second instance it will result in a quotient of k and a remainder of 1.

Now let us formulate a statement about the natural numbers and prove it. Following tradition, we refer to formal mathematical statements either as *theorems* or *propositions* or sometimes as *lemmas* or *corollaries*. A theorem is supposed to be an important statement that is the culmination of some development of significant ideas. A proposition is a statement of lesser intrinsic importance. Usually a lemma is of no intrinsic interest, but is needed as a step along the way to verifying a theorem or a proposition. Finally, a corollary is usually a direct consequence of a theorem or proposition. The corollary can often be a matter of considerable interest; but its proof should be a brief consequence of the the theorem or proposition in question.

Proposition: *The square of an even natural number is even.*
Proof: *Let us begin by using what we learned in an earlier lecture. We may reformulate our statement as "If n is even, then $n \cdot n$ is even." This statement makes a promise. Refer to the definition of "even" to see what that promise is:*

If n can be written as twice a natural number, then $n \cdot n$ can be written as twice a natural number.

The hypothesis of the assertion is that $n = 2 \cdot m$ for some natural number m. But then

$$n^2 = n \cdot n = (2m) \cdot (2m) = 4m^2 = 2(2m^2).$$

Our calculation shows that n^2 is twice the natural number $2m^2$. So n^2 is also even.

We have shown that the hypothesis that n is twice a natural number entails the conclusion that n^2 is twice a natural number. In other words, if n is even, then n^2 is even. That is the end of our proof. $\square$

Remark: What is the role of truth tables at this point? Why did we not use a truth table to verify our proposition? One *could* think of the statement that we are proving as the conjunction of infinitely many specific statements about concrete instances of the variable n; and then we could verify each one of those statements. But such a procedure is inelegant and, more importantly, impractical.

For our purposes, the truth table *tells us what we must do to construct a proof.* The truth table for $A \Rightarrow B$ shows that, if A is false, then there is nothing to check; whereas, if A is true, then we must show that B is true. That is just what we did in the proof of Proposition 2.2.3.

Most of our theorems are "for all" statements or "there exists" statements. In practice, it is not usually possible to verify them directly by use of a truth table.

Proposition: *The square of an odd natural number is odd.*

Proof: *We follow the paradigm laid down in the proof of the previous proposition.*

Assume that n is odd. Then $n = 2m + 1$ for some natural number m. But then

$$n^2 = n \cdot n = (2m+1) \cdot (2m+1) = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1.$$

We see that n^2 is $2m' + 1$, where $m' = 2m^2 + 2m$. In other words, according to our definition, n^2 is odd. □

Both of the proofs that we have presented are examples of "direct proof." A direct proof proceeds according to the statement being proved; for instance, if we are proving a statement about a square, then we calculate that square. If we are proving a statement about a sum, then we calculate that sum. Here are some additional examples:

Example

Prove that, if n is a positive integer, then the quantity $n^2 + 3n + 2$ is even.

Proof: Denote the quantity $n^2 + 3n + 2$ by K. Observe that

$$K = n^2 + 3n + 2 = (n+1)(n+2) .$$

Thus K is the product of two successive integers: $n + 1$ and $n + 2$. One of those two integers must be even. So it is a multiple of 2. Therefore K itself is a multiple of 2. Hence K must be even. □

Proposition: *The sum of two odd natural numbers is even.*

Proof: *Suppose that* p *and* q *are both odd natural numbers. According to the definition, we may write* $p = 2r + 1$ *and* $q = 2s + 1$ *for some natural numbers* r *and* s*. Then*

$$p + q = (2r + 1) + (2s + 1) = 2r + 2s + 2 = 2(r + s + 1).$$

We have realized $p + q$ *as twice the natural number* $r + s + 1$*. Therefore* $p + q$ *is even.* □

Remark: In some subjects, such as literary criticism or philosophy, it is common to reason by analogy, or to present ideas so that they sound good. If we did mathematics solely according to what sounds good, or what appeals intuitively, then we might reason as follows: "If the sum of two odd natural numbers is even then it must be that the sum of two even natural numbers is odd." This is incorrect. For instance, 4 and 6 are each even but their sum $4 + 6 = 10$ is *not* odd.

Intuition definitely plays an important role in the development of mathematics, but all assertions in mathematics must, in the end, be proved by rigorous methods.

Example

Prove that the sum of an even integer and an odd integer is odd.

Proof: An even integer e is divisible by 2, so may be written in the form $e = 2m$, where m is an integer. An odd integer o has remainder 1 when divided by 2, so may be written in the form $o = 2k + 1$, where k is an integer. The sum of these is

$$e + o = 2m + (2k + 1) = 2(m + k) + 1 .$$

Thus we see that the sum of an even and an odd integer will have remainder 1 when it is divided by 2. As a result, the sum is odd. □

Proposition: *The sum of two even natural numbers is even.*
Proof: *Let* $p = 2r$ *and* $q = 2s$ *both be even natural numbers. Then*

$$p + q = 2r + 2s = 2(r + s).$$

We have realized $p + q$ *as twice a natural number. Therefore we conclude that* $p + q$ *is even.* □

Proposition: *Let n be a natural number. Then either $n > 6$ or $n < 9$.*

Proof: *If you draw a picture of a number line then you will have no trouble convincing yourself of the truth of the assertion. What we want to learn here is to organize our thoughts so that we may write down a rigorous proof.*

Our discussion of the connective "or" in Section 1.3 will now come to our aid. Fix a natural number n. If $n > 6$ then the 'or' statement is true and there is nothing to prove. If $n \not> 6$, then the truth table teaches us that we must check that $n < 9$. But the statement $n \not> 6$ means that $n \leq 6$ so we have

$$n \leq 6 < 9.$$

That is what we wished to prove. □

Example

Prove that every even integer may be written as the sum of two odd integers.

Proof: Let the even integer be $K = 2m$, for m an integer. If m is odd then we write

$$K = 2m = m + m$$

and we have written K as the sum of two odd integers. If, instead, m is even, then we write

$$K = 2m = (m - 1) + (m + 1) .$$

Since m is even then both $m - 1$ and $m + 1$ are odd. So again we have written K as the sum of two odd integers. □

Example: Prove the Pythagorean theorem.

Proof: The Pythagorean theorem states that $c^2 = a^2 + b^2$, where a and b are the legs of a right triangle and c is its hypotenuse.

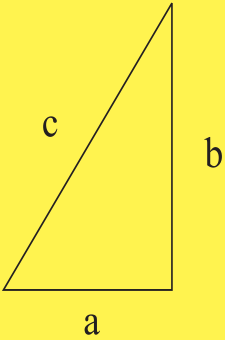

Figure: A right triangle.

Consider now the arrangement of four triangles and a square shown in Figure 2.2. Each of the four triangles is a copy of the original triangle. We see that each side of the all-encompassing square is equal to c. So the area of that square is c^2. Now each of the component triangles has base a and height b. So each such triangle has area $ab/2$. And the little square in the middle has side $b - a$. So it has area $(b - a)^2 = b^2 - 2ab + a^2$. We write the total area as the sum of its component areas:

$$c^2 = 4 \cdot \left[\frac{ab}{2} \right] + [b^2 - 2ab + a^2] = a^2 + b^2 \, .$$

That is the desired equality. □

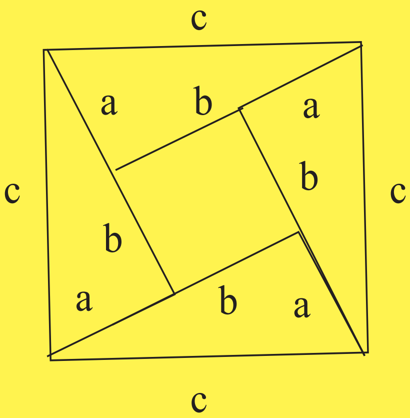

Figure: The Pythagorean theorem.

In this section and the next two, we are concerned with form rather than substance. We are not interested in proving anything profound, but rather in showing you what a proof looks like. Later in the book we shall consider some deeper mathematical ideas and correspondingly more profound proofs.