# Math 310
# October 5, 2020 Lecture

Steven G. Krantz

October 1, 2020

Figure: This is your instructor.

### Definition

We let $\emptyset$ denote the set that contains no elements. That is, $\forall x, x \notin \emptyset$. We call $\emptyset$ the *empty set*.

It may seem strange to consider a set with no elements. But this set arises very naturally in many mathematical contexts. For example, consider the set

$$S = \{x \in \mathbb{R} : x^2 < 0\} .$$

There are no real numbers with negative square. So there are no elements in this set. It is useful to be able to write $S = \emptyset$.

Example

If $S$ is any set, then $\emptyset \subset S$. To see this, notice that the statement "If $x \in \emptyset$ then $x \in S$" *must* be true because the hypothesis $x \in \emptyset$ is false. [Check the truth table for "if-then" statements.] This verifies that $\emptyset \subset S$.

Example

Let $S = \{x \in \mathbb{N} : x + 2 \geq 19 \text{ and } x < 3\}$. Then $S$ is a sensible set. There are no internal contradictions in its definition. But $S = \emptyset$. There are no elements in $S$.

Definition

Let $S$ and $T$ be sets. We say that $x$ is an element of $S \cap T$ if both $x \in S$ *and* $x \in T$. It is useful and enlightening to write

$$x \in S \cap T \Longleftrightarrow x \in S \wedge x \in T .$$

This relates our new set-theoretic idea to basic concepts of logic that we learned in Chapter 1.

We say that $x$ is an element of $S \cup T$ if either $x \in S$ *or* $x \in T$. Again, it is useful to relate the new idea to basic logic by writing

$$x \in S \cup T \Longleftrightarrow x \in S \vee x \in T \, .$$

We call $S \cap T$ the *intersection* of the sets $S$ and $T$. We call $S \cup T$ the *union* of the sets $S$ and $T$.

## Example

Let $S = \{x \in \mathbb{N} : 2 < x < 9\}$ and $T = \{x \in \mathbb{N} : 5 \leq x < 14\}$. Then $S \cap T = \{x \in \mathbb{N} : 5 \leq x < 9\}$, for these are the points common to both sets. And $S \cup T = \{x \in \mathbb{N} : 2 < x < 14\}$, for these are the points that are either in $S$ or in $T$ or in both. Draw a diagram on a real line to help you understand this example.

**Remark:** Observe that the use of "or" in the definition of set union justifies our decision to use the "inclusive 'or' " rather than the "exclusive 'or'" in mathematics.

Example

Let $S = \{x \in \mathbb{N} : 1 \leq x \leq 5\}$ and $T = \{x \in \mathbb{N} : 8 < x \leq 12\}$. Then $S \cap T = \emptyset$, for the sets $S$ and $T$ have no elements in common. On the other hand,
$S \cup T = \{x \in \mathbb{N} : 1 \leq x \leq 5 \text{ or } 8 < x \leq 12\}$. Draw a diagram on a real line to help you understand this example.

We may consider the intersection of more than two sets. For example, $x \in S \cap T \cap U$ means that (simultaneously) $x \in S$, $x \in T$, and $x \in U$. In fact one can consider the intersection of any number of sets—even an infinite number.

### Example

Let $S = \{1, 2, 3\}$, $T = \{2, 3, 4\}$, and $U = \{3, 4, 1\}$. Then

$$S \cap T = \{2, 3\}\ ,\ T \cap U = \{3, 4\}\ ,\ U \cap S = \{1, 3\}\ ,\ S \cap T \cap U = \{3\}\ .$$

### Example

Let $S = \{1, 2\}$, $T = \{2, 3\}$, and $U = \{3, 1\}$. Then $S \cap T \cap U = \emptyset$. In other words, the three sets have no elements in common. But $S \cap T = \{2\} \neq \emptyset$, $T \cap U = \{3\} \neq \emptyset$, and $U \cap S = \{1\} \neq \emptyset$. It is common to say that $S, T, U$ are *disjoint sets*.

But $S$, $T$, and $U$ are *not pairwise disjoint*. The phrase "pairwise disjoint" means that no two of the sets have any elements in common. For instance, $A = \{1, 2\}$, $B = \{3, 4\}$, and $C = \{5, 6\}$ *are* pairwise disjoint.

### Definition

Let $S$ and $T$ be sets. We say that $x \in S \setminus T$ if both $x \in S$ and $x \notin T$. We may write this logically as

$$x \in S \setminus T \iff x \in S \wedge x \notin T .$$

We call $S \setminus T$ the *set-theoretic difference* of $S$ and $T$.

Example

Let $S = \{x \in \mathbb{N} : 2 < x < 7\}$ and $T = \{x \in \mathbb{N} : 5 \leq x < 10\}$.
Then $S \setminus T = \{x \in \mathbb{N} : 2 < x < 5\}$ and
$T \setminus S = \{x \in \mathbb{N} : 7 \leq x < 10\}$.

**Definition**

Suppose that we are studying subsets of a fixed set $X$. If $S \subset X$, then we use the symbol ${}^cS$ to denote $X \setminus S$. In this context, we sometimes refer to $X$ as the *universal set*. We call ${}^cS$ the *complement* of $S$ (in $X$). We may write

$$x \in {}^cS \iff x \in X \wedge x \notin S .$$

### Example

Let $\mathbb{N}$ be the universal set. Let $S = \{x \in \mathbb{N} : 3 < x \leq 20\}$. Then

$$^{c}S = \{x \in \mathbb{N} : 1 \leq x \leq 3\} \cup \{x \in \mathbb{N} : 20 < x\}.$$

**Proposition:** *Let $X$ be the universal set and $S \subset X$, $T \subset X$. Then*

*(a)* ${}^c\big(S \cup T\big) = {}^cS \cap {}^cT$;
*(b)* ${}^c\big(S \cap T\big) = {}^cS \cup {}^cT$.

**Proof:** We shall present this proof in detail since it is a good exercise in understanding both our definitions and our method of proof (and also a good exercise with logic).

We begin with the proof of (a). It is often best to treat the proof of the equality of two sets as two separate proofs of containment. [This is why Proposition 3.2.5 is important.] That is what we now do.

Let $x \in {}^c(S \cup T)$. Then, by definition, $x \notin (S \cup T)$. Thus $x$ is neither an element of $S$ nor an element of $T$. So both $x \in {}^cS$ and $x \in {}^cT$. Hence $x \in {}^cS \cap {}^cT$. We conclude that ${}^c(S \cup T) \subset {}^cS \cap {}^cT$. Conversely, if $x \in {}^cS \cap {}^cT$, then $x \notin S$ and $x \notin T$. Therefore $x \notin (S \cup T)$. As a result, $x \in {}^c(S \cup T)$. Thus ${}^cS \cap {}^cT \subset {}^c(S \cup T)$. Summarizing, we have ${}^c(S \cup T) = {}^cS \cap {}^cT$.

The proof of part (b) is similar, but we include it for practice. Let $x \in {}^c(S \cap T)$. Then, by definition, $x \notin (S \cap T)$. Thus $x$ is not both an element of $S$ and an element of $T$. So either $x \in {}^cS$ or $x \in {}^cT$. Hence $x \in {}^cS \cup {}^cT$. We conclude that ${}^c(S \cap T) \subset {}^cS \cup {}^cT$. Conversely, if $x \in {}^cS \cup {}^cT$, then either $x \notin S$ or $x \notin T$. Therefore $x \notin (S \cap T)$. As a result, $x \in {}^c(S \cap T)$. Thus ${}^cS \cup {}^cT \subset {}^c(S \cap T)$. Summarizing, we have ${}^c(S \cap T) = {}^cS \cup {}^cT$. □

The two formulas in the last proposition are often referred to as de Morgan's Laws. Compare them with de Morgan's laws for $\vee$ and $\wedge$ in Section 1.3 and also with the de Morgan's Laws that we discussed in Section 1.7.