Math 310
October 7, 2020 Lecture

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Figure: This is your instructor.

Venn Diagrams

We sometimes use a *Venn diagram* to aid our understanding of set-theoretic relationships. In a Venn diagram, a set is represented as a region in the plane (for convenience, we use rectangles). The intersection $A \cap B$ of two sets A and B is the region common to the two domains (we have shaded that region in the figure):

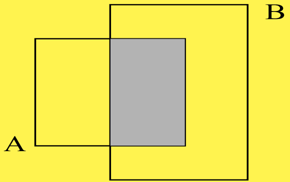

Figure: A Venn diagram showing $A \cap B$.

Now let A, B, and C be three sets. The Venn diagram in the next figure makes it easy to see that
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

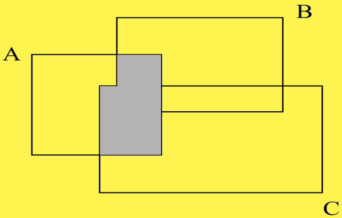

Figure: A Venn diagram that illustrates $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

The Venn diagram in this last figure illustrates the fact that

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$$

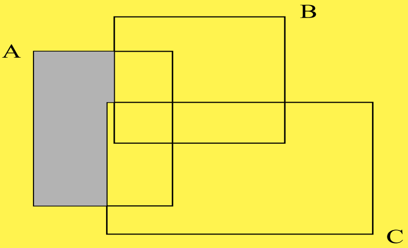

Figure: A Venn diagram that illustrates $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

A Venn diagram is not a proper substitute for a rigorous mathematical proof. However, it can go a long way toward guiding our intuition.

Further Ideas in Elementary Set Theory

Now we learn some new ways to combine sets.

Definition

Let S and T be sets. We define $S \times T$ to be the set of all ordered pairs (s, t) such that $s \in S$ and $t \in T$. The set $S \times T$ is called the *set-theoretic product* (or sometimes just the *product*) of S and T.

It is worth pausing a moment to consider this last definition. Strictly speaking, it is not entirely satisfactory because we have not defined "ordered pair." Any attempt to do so using phrases like "first element" and "second element" seems to lead to more questions than answers. A rigorous and elegant way to define the ordered pair (a, b) is that it is equal to the set $\{\{a\}, \{a, b\}\}$. This definition exhibits directly that the ordered pair contains both the elements a and b and that the element a is distinguished.

Example

Let $S = \{1, 2, 3\}$ and $T = \{a, b\}$. Then

$$S \times T = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}.$$

It is no coincidence that, in the last example, the set S has 3 elements, the set T has 2 elements, and the set $S \times T$ has $3 \times 2 = 6$ elements. In fact one can prove that if S has k elements and T has ℓ elements, then $S \times T$ has $k \cdot \ell$ elements. Exercise 3.20 asks you to prove this assertion by mathematical induction on k.

Notice that $S \times T$ is a different set from $T \times S$. With S and T as in the last example,

$$T \times S = \{(a,1),(b,1),(a,2),(b,2),(a,3),(b,3)\}.$$

The phrase "ordered pair" means that the pair $(a,1)$, for example, is distinct from the pair $(1,a)$.

If S is a set then the *power set* of S is the set of all subsets of S. We denote the power set by $\mathcal{P}(S)$. We may write

$$X \in \mathcal{P}(S) \Longleftrightarrow X \subset S\,.$$

Example

Let $S = \{1, 2, 3\}$. Then

$$\mathcal{P}(S) = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}, \emptyset\}.$$

If the concept of power set is new to you, then you might have been surprised to see $\{1, 2, 3\}$ and $\emptyset$ as elements of the power set. But they are both subsets of S, and they must be listed.

Proposition: *Let $S = \{s_1, \ldots, s_j\}$ be a set. Then $\mathcal{P}(S)$ has 2^j elements.*

Proof: We prove the assertion by mathematical induction on j.

(a) The assertion $P(j)$ is this:

The power set of a set with j elements has 2^j elements.

(b) $P(1)$ is true. In this case, $S = \{s_1\}$ and $\mathcal{P}(S) = \{\{s_1\}, \emptyset\}$. Notice that S has $j = 1$ element and $\mathcal{P}(S)$ has $2^j = 2$ elements.

(c) $P(j) \Rightarrow P(j+1)$. Assume that any set of the form $S = \{s_1, \ldots, s_j\}$ has power set with 2^j elements. Now let $T = \{t_1, \ldots, t_j, t_{j+1}\}$. Consider the subset $T' = \{t_1, \ldots, t_j\}$ of T. Then $\mathcal{P}(T)$ certainly contains $\mathcal{P}(T')$ (that is, every subset of T' is also a subset of T). But it also contains each of the sets that is obtained by adjoining the element t_{j+1} to a subset of T'. Thus the total number of subsets of T is

$$2^j + 2^j = 2^{j+1} .$$

Notice that we have indeed counted all subsets of T, since any subset either contains t_{j+1} or it does not.

Thus, assuming the validity of our assertion for j, we have proved its validity for $j+1$. That completes our mathematical induction and the proof of the proposition. $\square$

We have seen that the operation of set-theoretic product corresponds to the arithmetic product of natural numbers. And now we have seen that the operation of taking the power set corresponds to exponentiation. In Section 4.3 we shall use the concept of function to unify all of these ideas.