Math 310
October 9, 2020 Lecture

Steven G. Krantz

October 2, 2020

Figure: This is your instructor.

Indexing and Extended Set Operations

Frequently we wish to manipulate infinitely many sets. Perhaps we will take their intersection or union. We require suitable notation to perform these operations.

If $S_1, S_2, \ldots$ are sets, then we define

$$\bigcup_{j=1}^{\infty} S_j \equiv \{x : \exists j \text{ such that } x \in S_j\}.$$

Similarly, we define

$$\bigcap_{j=1}^{\infty} S_j \equiv \{x : \forall j, x \in S_j\}.$$

Notice that we employ the common mathematical notation $\equiv$ to mean "is defined to be." Other texts use the notation $\stackrel{\text{def}}{=}$ or $=:$ or $\doteq$.

Example

Let $\mathbb{Q}$ be the rational numbers and let $S_j = \{x \in \mathbb{Q} : 0 < x < 1 + 1/j\}$, $j = 1, 2, \dots$. Let us describe $\cup_{j=1}^{\infty} S_j$ and $\cap_{j=1}^{\infty} S_j$.

Notice that $S_1 \supset S_j$ for every j, hence $\cup_{j=1}^{\infty} S_j = S_1 = \{x \in \mathbb{Q} : 0 < x < 2\}$.

Next, notice that, if $x \in \mathbb{Q}$ and $x > 1$, then if we select $j > 1/(x-1)$ then $x \notin S_j$. It follows that $x \notin \cap_{j=1}^{\infty} S_j$. On the other hand, $\{x \in \mathbb{Q} : 0 < x \leq 1\} \subset S_j$ for every j. It follows that $\cap_{j=1}^{\infty} S_j = \{x \in \mathbb{Q} : 0 < x \leq 1\}$.

Example

It is entirely possible for *nested*, nonempty sets to have empty intersection. Let $S_j = \{x \in \mathbb{Q} : 0 < x < 1/j\}$. Certainly each S_j is nonempty, for it contains the point $1/(2j)$. In fact each S_j has infinitely many elements. Next, $S_1 \supset S_2 \supset \cdots$. Finally, for any positive integer M,

$$\bigcap_{j=1}^{M} S_j = S_M \neq \emptyset.$$

However,

$$\bigcap_{j=1}^{\infty} S_j = \emptyset.$$

To verify this last assertion, notice that, if $x > 0$ and $j > 1/x$, then $x \notin S_j$ hence $x \notin \cap_{j=1}^{\infty} S_j$. However, if $x \leq 0$, then x is not an element of any S_j. As a result, no x lies in the intersection. The intersection is empty.

In the examples given thus far, the "index set" has been the natural numbers. That is, we let the index j range over $\{1, 2, \dots\}$. It is frequently useful to use a larger index set, such as the real numbers or some unspecified index set. Usually we specify an index set with the letter A and we denote a specific index by $\alpha \in A$.

Example

For each real number α we let $S_\alpha = \{x \in \mathbb{R} : \alpha \leq x < \alpha + 1\}$. Thus each S_α is an "interval" of real numbers, and we may speak of

$$\bigcup_{\alpha \in A} S_\alpha \equiv \{x : \exists \alpha, x \in S_\alpha\}$$

and

$$\bigcap_{\alpha \in A} S_\alpha \equiv \{x : \forall \alpha, x \in S_\alpha\} .$$

For the sets S_α that we have specified,

$$\bigcap_{\alpha \in A} S_\alpha = \emptyset.$$

This is so because, if $x \in \mathbb{R}$, then $x \notin S_{x+1}$ hence certainly $x \notin \cap_\alpha S_\alpha$.

On the other hand,

$$\bigcup_{\alpha \in A} S_\alpha = \mathbb{R}$$

since every real x lies in $S_{x-1/2}$.

Proposition: *Fix a universal set X. Let A be an index set and, for each $\alpha \in A$, let S_α be a subset of X. Then*

(a) $^c\left(\bigcap_{\alpha\in A} S_\alpha\right) = \bigcup_{\alpha\in A} {}^c S_\alpha$;

(b) $^c\left(\bigcup_{\alpha\in A} S_\alpha\right) = \bigcap_{\alpha\in A} {}^c S_\alpha$.

Proof: The proof is similar to that of earlier versions of de Morgan's Laws. We leave the details to the exercises at the end of the chapter. □

Further properties of intersection and union over arbitrary index sets are explored in the exercises. These are some of the most important exercises in the chapter.