Math 310
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Figure: This is your instructor.

In this section we discuss the concept of ordering a set. There are many different types of orderings; we shall concentrate on just a few of these.

Definition Let S be a set and $\mathcal{R}$ a relation on S. We call $\mathcal{R}$ a *partial ordering* on S if it satisfies the following properties:

(a) For all $x \in S$, $(x, x) \in \mathcal{R}$;

(b) If $x, y \in S$ and both $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$, then $x = y$;

(c) If $x, y, z \in S$ and both $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$, then $(x, z) \in \mathcal{R}$.

Not surprisingly, we refer to property **(a)** as *reflexivity*, to property **(b)** as *anti-symmetry*, and to property **(c)** as *transitivity*.

Example Let S be the power set of $\mathbb{R}$—the set of all sets of real numbers. If $A, B \in S$, then let us say that $(A, B) \in \mathcal{R}$ if $A \subset B$, that is, if every element of A is also an element of B. Check that the three axioms for a partial ordering are satisfied by our relation $\mathcal{R}$. We usually write this relation in the binary form $A \subset B$.

A noteworthy feature of a *partial ordering* (as opposed to a total ordering, to be discussed below) is that not every two elements of the set S need be comparable. The last example illustrates this point: if $A = \{x \in \mathbb{R} : 1 < x < 4\}$ and $B = \{x \in \mathbb{R} : 2 < x < 9\}$, then both $A, B \in S$ yet neither $(A, B) \in \mathcal{R}$ nor $(B, A) \in \mathcal{R}$.

Definition Let S be a set and $\mathcal{R}$ a relation on S. We call $\mathcal{R}$ a *simple ordering* on S if it satisfies the following properties:

(a) If $x, y \in S$ and both $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$, then $x = y$;

(b) If $x, y, z \in S$ and both $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$, then $(x, z) \in \mathcal{R}$;

(c) If $x, y \in S$ are distinct, then either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$.

Observe that property **(c)** distinguishes a simple ordering from a partial ordering.

As before, we refer to **(a)** and **(b)** as the properties of anti-symmetry and transitivity, respectively. We refer to property **(c)** as *strong connectivity*. A simple ordering is sometimes also called a *total ordering*.

Example Let $S = \mathbb{R}$, the real numbers. Let us say that $(x, y) \in \mathcal{R}$ if $y - x \geq 0$. It is straightforward to verify that properties **(a)**, **(b)**, **(c)** of a simple ordering hold for this relation. We usually write this relation in the binary form $x \leq y$.

Definition Let us say that a set S is *strictly simply ordered* by a relation $\mathcal{R}$ if properties **(b)** and **(c)** of the last definition hold but property **(a)** is replaced by

(a′) If $x, y \in S$ and $(x, y) \in \mathcal{R}$ then $(y, x) \notin \mathcal{R}$.

Example The real numbers $S = \mathbb{R}$ are strictly simply ordered by the binary relation $<$.

Suppose that S is a set that is equipped with a strict, simple ordering $\mathcal{R}$. Let $A \subset S$. An element $a \in A$ is called *minimal* (for this ordering) in A if $(a, x) \in \mathcal{R}$ for all $x \in A$, $x \neq a$. We also sometimes call a the *least* element of A. It is clear from this definition that the minimal element is unique if it exists.

Example Let $S = \mathbb{N}$, the natural numbers. If $m, n \in \mathbb{N}$ then we say that $m\mathcal{R}n$ if $m < n$. This is a strict, simple ordering. Then $T = \{8, 4, 9, 17, 3\}$ is a subset of $\mathbb{N}$, and 3 is the minimal element of T.

Definition Let us say that a strict, simple ordering $\mathcal{R}$ *well orders* a set S if each nonempty subset $A \subset X$ has a minimal element.

Example The usual ordering $<$ well orders the natural numbers. That is to say, each nonempty subset of the natural numbers $\{1, 2, 3, \dots\}$ has a minimal element. This statement is intuitively clear, but proving it quickly leads to deep and difficult questions about the foundations of mathematics. We shall treat the issue in greater detail in a later lecture.

The ordering $<$ does *not* well order the integers $\mathbb{Z}$, nor does it well order the real numbers $\mathbb{R}$. For example $\mathcal{E}$, the even numbers, is a subset of $\mathbb{Z}$ and is also a subset of $\mathbb{R}$; but $\mathcal{E}$ certainly has no least element. The subset $S = \{x \in \mathbb{R} : 0 < x < 1\}$ (with the usual ordering on $\mathbb{R}$) has no least element.

In fact, one way to well order the integers is to construct a one-to-one correspondence between the integers and the natural numbers and then to pull the natural ordering from the natural numbers back to the integers by way of this correspondence. This gives a well ordering of the integers, but it is certainly not the standard ordering. We shall say more about this technique in a later lecture.

It is impossible to explicitly specify a well ordering for the real numbers $\mathbb{R}$, although such a well ordering *does* exist. In fact absolutely any set can be well ordered (although it is often not at all clear how to actually perform the ordering). This matter is intimately connected with the so-called Axiom of Choice. We shall discuss that axiom in a future lecture.

We shall have more to say about orderings in later lectures.