Math 310
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Figure: This is your instructor.

Functions

In more elementary mathematics courses, we define a function as follows: Let S and T be sets. A function f from S to T is a rule that assigns to each element of S a unique element of T.

This definition is problematic. The main difficulty is the use of the words "rule" and "assign." For instance, let $S = T = \mathbb{Z}$. Consider

$$f(x) = \begin{cases} x^2 & \text{if there is life as we know it on Mars} \\ 3x - 5 & \text{if there is not life as we know it on Mars.} \end{cases}$$

Is this a function? Can what we see on the right be considered a rule? Do we have to wait until we have found life on Mars before we can consider this a function?

More significantly in practice, thinking of a function as a rule is extremely limiting. The functions

$$\begin{aligned} f(x) &= x^3 - 3x + 1 \\ g(x) &= \sin x \\ h(x) &= \frac{\ln x}{x^2 + 4} \end{aligned}$$

are inarguably given by rules. But open up your newspaper and look on the financial page at the graph of the Gross National Product. This is certainly the graph of a function, but what "rule" describes it?

It is best in advanced mathematics to have a way to think about functions that avoids subjective words like “rule” and “assign.” This is the motivation for our next definition.

Definition Let S and T be sets. A *function* f from S to T is a relation on S and T such that

(i) Every $s \in S$ is in the domain of f;

(ii) If $(s, t) \in f$ and $(s, u) \in f$ then $t = u$.

Of course we refer to S and T as the domain and the range, respectively,[1] of f. Condition (i) mandates that each element s of S is associated to *some* element of T. Condition (ii) mandates, in a formal manner, that each element s of S is associated to *only one* member of T. Notice, however, that the definition neatly sidesteps the notions of "assign" or "rule." Now look back at our "Mars" definition and decide whether it is a function.

[1] We note, once again, that some sources use the word "codomain" instead of "range."

We shall frequently speak of the *image* of a given function f from S to T. This just means the set that is the image of f when it is thought of as a relation. It is the set of elements $t \in T$ such that there is an $s \in S$ with $(s, t) \in f$.

Example Let $S = \{1, 2, 3\}$ and $T = \{a, b, c\}$. Set

$$f = \{(1, a), (2, a), (3, b)\}.$$

This is a function, for it satisfies the properties set down in the definition. Given the way that you are accustomed to writing functions in earlier courses, you might find it helpful to view this function as

$$\begin{aligned} f(1) &= a \\ f(2) &= a \\ f(3) &= b. \end{aligned}$$

Notice that each element $1, 2, 3$ of the domain is “assigned” to one and only one element of the range. However, the definition of function allows the possibility that two different elements of the domain be assigned to the same range element. Observe that, for this function, the image is $\{a, b\}$ while the range is $\{a, b, c\}$.

Example Let $S = \{1, 2, 3\}$ and $T = \{a, b, c, d, e\}$. Set

$$f = \{(1, b), (2, c), (3, e)\}.$$

This is a function, for it satisfies the properties set down in the definition. Notice that each element of the domain S is used once and only once. However, not all elements of the range are used. According to the definition of function, this is allowed.

Example Let $S = \{1, 2, 3, 4, 5\}$ and let $T = \{a, b, c\}$. This time there are more elements in S than there are in T. Nonetheless,

$$f = \{(1, a), (2, a), (3, b), (4, b), (5, c)\}$$

is a function. It repeats values, but it definitely satisfies the definition.

Definition Let f be a function with domain S and range T. We often write such a function as $f : S \to T$. We say that f is *one-to-one* or *injective* if, whenever $(s, t) \in f$ and $(s', t) \in f$, then $s = s'$. We sometimes refer to such a mapping as an *injection*. We also call such a map *injective* or *univalent*.

Compare this new definition with the definition of function. The new condition is similar to condition (ii) for functions. But it is *not* the same. We are now mandating that no two domain elements be associated with the same range element.

Example Let $S = T = \mathbb{R}$ and let f be the set of all ordered pairs $\{(x, x^2) : x \in \mathbb{R}\}$. We may also write this function as

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^2 \end{aligned}$$

or as $f(x) = x^2$.

It is easy to verify that f satisfies the definition of function. However, both of the ordered pairs $(-2, 4)$ and $(2, 4)$ are in f (in other words, $f(-2) = 4 = f(2)$) so that f is *not* one-to-one.

Example Let $S = T = \mathbb{R}$ and let f be the function $f(x) = x^3 + x - 5$. Then $f'(x) = 3x^2 + 1 > 0$ for every x. Therefore f is a strictly increasing function. In particular, if $s < t$, then $f(s) < f(t)$ so that $f(s) \neq f(t)$. It follows that the function f *is* one-to-one.

Definition Let f be a function with domain S and range T. If, for each $t \in T$ there is an $s \in S$ such that $f(s) = t$, then we say that f is *onto* or *surjective*. We sometimes refer to such a mapping as a *surjection*. Notice that a function is onto precisely when its image equals its range.

Example Let $f(x) = x^2$ be the function from Example 4.3.6. Recall from that example that $S = T = \mathbb{R}$. The point $t = -1 \in T$ has the property that there is no $s \in S$ such that $f(s) = t$. As a result, this function f is *not onto*.

Example Let $S = \mathbb{R}$, $T = \{x \in \mathbb{R} : 1 \leq x < \infty\}$. Let $g : S \to T$ be given by $g(x) = x^2 + 1$. Then for each $t \in T$ the number $s = +\sqrt{t-1}$ makes sense and lies in S. Moreover, $g(s) = t$. It follows that this function g is surjective. However, g is not injective.

Example Have another look at the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. We have already noted, in an earlier example, that this function is not onto. But if we restrict the range to $T = \{y \in \mathbb{R} : y \geq 0\}$, so that $f : \mathbb{R} \to T$, then it is easy to verify that the function is now onto. In other words, every nonnegative real number has a square root.

Combining Functions

There are several elementary operations that allow us to combine functions in useful ways. In this section, and from now on, we shall (whenever possible) write our functions in the form

$$f(x) = \text{(formula)}$$

for the sake of clarity. However, we must keep in mind, and we shall frequently see, that many functions *cannot* be expressed with an elegant formula.