Figure: This is your instructor.
Definition Let $f$ and $g$ be functions with the same domain $S$ and the same range $T$. Assume that $T$ is a set in which the indicated arithmetic operations (below) makes sense. Then we define

(a) $(f + g)(x) = f(x) + g(x)$;
(b) $(f - g)(x) = f(x) - g(x)$;
(c) $(f \cdot g)(x) = f(x) \cdot g(x)$;
(d) $(f/g)(x) = f(x)/g(x)$ provided that $g(x) \neq 0$. 
Notice that, in each of (a)–(d), we are defining a new function—either \( f + g \) or \( f - g \) or \( f \cdot g \) or \( f/g \)—in terms of the component functions \( f \) and \( g \). For practice, we shall express (a) in the language of ordered pairs. We ask you to do likewise with (b), (c), (d) in one of the exercises.

Let us consider part (a) in detail. Now \( f \) is a collection of ordered pairs in \( S \times T \) that satisfy the conditions for a function, and so is \( g \). The function \( f + g \) is given by

\[
(f + g) = \{(s, t + t') : (s, t) \in f, (s, t') \in g\}.
\]
Expressing the other combinations of \( f \) and \( g \) is quite similar, and you should be sure to do the corresponding exercise in the book.

**Example** Let \( S = T = \mathbb{R} \). Define

\[
f(x) = x^3 - x \quad \text{and} \quad g(x) = \sin(x^2).
\]

Let us calculate \( f + g \), \( f - g \), \( f \cdot g \), \( f/g \).

Now

\[
(f + g)(x) = (x^3 - x) + \sin(x^2)
\]
\[
(f - g)(x) = (x^3 - x) - \sin(x^2)
\]
\[
(f \cdot g)(x) = (x^3 - x) \cdot [\sin(x^2)]
\]
\[
(f/g)(x) = (x^3 - x)/[\sin(x^2)] \quad \text{provided} \ x \neq \pm \sqrt{k\pi}, \]
\[
\quad k \in \{0, 1, 2, \ldots \}.
\]
A more interesting, and more powerful, way to combine functions is through functional composition. Incidentally, in this discussion we will see the value of good mathematical notation.

**Definition** Let \( f : S \rightarrow T \) be a function, and let \( g : T \rightarrow U \) be a function. Then we define, for \( s \in S \), the composite function

\[
(g \circ f)(s) = g(f(s)).
\]

We call \( g \circ f \) the *composition* of the functions \( g \) and \( f \).
Notice in this definition that the right-hand side of (*) always makes sense because of the way that we have specified the domain and range of the component functions $f$ and $g$. In particular, we must have image $f \subset \text{domain } g$ in order for the composition to make sense.
Example Let $f : \mathbb{R} \to \{x \in \mathbb{R} : x \geq 0\}$ be given by $f(x) = x^4 + x^2 + 6$ and $g : \{x \in \mathbb{R} : x \geq 0\} \to \mathbb{R}$ be given by $g(x) = \sqrt{x} - 4$. Notice that $f$ and $g$ fit the paradigm specified in the definition of composition of functions. Then

$$(g \circ f)(x) = g(f(x))$$

$$= g(x^4 + x^2 + 6)$$

$$= \sqrt{x^4 + x^2 + 6} - 4.$$
Notice that $f \circ g$ also makes sense and is given by

$$(f \circ g)(x) = f(g(x))$$

$$= f(\sqrt{x} - 4)$$


It is important to understand that $f \circ g$ and $g \circ f$, when both make sense, will generally be different.
It is a good exercise in the ideas of this chapter to express the notion of functional composition in the language of ordered pairs. Thus let $f : S \rightarrow T$ be a function and $g : T \rightarrow U$ be a function. Then $f$ is a subset of $S \times T$ and $g$ is a subset of $T \times U$, both satisfying the two standard conditions for function. Now $g \circ f$ is a set of ordered pairs specified by
\[ g \circ f = \{(s, u) : s \in S, u \in U, \text{ and } \exists t \in T \text{ such that } (s, t) \in f \text{ and } (t, u) \in g \}. \]

Take a moment to verify that this equation is consistent with the definition of functional composition that we gave earlier. Further note that \( g \circ f \) is a set of ordered pairs from \( S \times U \).
Example Let \( f : \mathbb{R} \to \mathbb{R} \) be given by \( f(x) = \sin x^5 \) and let \( g : \{x \in \mathbb{R} : x \geq 1\} \to \mathbb{R} \) be given by \( \sqrt[4]{x - 1} \). We cannot consider \( g \circ f \) because the range of \( f \) (namely, the set \([-1, 1]\)) does not lie in the domain of \( g \). However, \( f \circ g \) does make sense because the range of \( g \) lies in the domain of \( f \). And

\[
(f \circ g)(x) = \sin \left[ (x - 1)^{5/4} \right].
\]
Definition Let $S$ and $T$ be sets. Let $f : S \to T$ and $g : T \to S$. We say that $f$ and $g$ are \textit{mutually inverse} provided that both $(f \circ g)(t) = t$ for all $t \in T$ and $(g \circ f)(s) = s$ for all $s \in S$. We write $g = f^{-1}$ or $f = g^{-1}$. We refer to the functions $f$ and $g$ as \textit{invertible}; we call $g$ the \textit{inverse} of $f$ and $f$ the \textit{inverse} of $g$. 
Example Let \( f : \mathbb{R} \to \mathbb{R} \) be given by \( f(x) = x^3 - 1 \) and \( g : \mathbb{R} \to \mathbb{R} \) be given by \( g(x) = \sqrt[3]{x + 1} \). Then

\[
(f \circ g)(x) = \left[ \sqrt[3]{x + 1} \right]^3 - 1 \\
= (x + 1) - 1 \\
= x
\]

and

\[
(g \circ f)(x) = \sqrt[3]{(x^3 - 1) + 1} \\
= \sqrt[3]{x^3} \\
= x
\]

for all \( x \). Thus \( g = f^{-1} \) (or \( f = g^{-1} \)).
The idea of inverse function lends itself particularly well to the notation of ordered pairs. For $f : S \to T$ is inverse to $g : T \to S$ (and vice versa) provided that for every ordered pair $(s, t) \in f$ there is an ordered pair $(t, s) \in g$ and conversely.
Not every function has an inverse. For instance, let $f : S \to T$. Suppose that $f(s) = t$ and also that $f(s') = t$ with $s \neq s'$ (in other words, suppose that $f$ is not one-to-one). If $g : T \to S$, then $g(f(s)) = g(t) = g(f(s'))$ so it cannot be that both $g(f(s)) = s$ and $g(f(s')) = s'$. In other words, $f$ cannot have an inverse. We conclude that a function that does have an inverse must be one-to-one.
On the other hand, suppose that $t \in T$ has the property that there is no $s \in S$ with $f(s) = t$ (in other words, suppose that $f$ is not onto). Then, in particular, it could not be that $f(g(t)) = t$ for any function $g : T \to S$. So $f$ could not be invertible. We conclude that a function that does have an inverse must be onto.
Example Let \( f : \mathbb{R} \rightarrow \{ x \in \mathbb{R} : x \geq 0 \} \) be given by \( f(x) = x^2 \). Then \( f \) is onto, but \( f \) is not one-to-one. It follows that \( f \) cannot have an inverse. And indeed it does not, for any attempt to produce an inverse function runs into the ambiguity that every positive number has two square roots.

Let \( f : \{ x \in \mathbb{R} : x \geq 0 \} \rightarrow \mathbb{R} \) be given by \( f(x) = x^2 \). Then \( f \) is one-to-one but \( f \) is not onto. There certainly is a function \( g : \mathbb{R} \rightarrow \{ x \in \mathbb{R} : x \geq 0 \} \) such that \((g \circ f)(x) = x\) for all \( x \in \{ x \in \mathbb{R} : x \geq 0 \} \) (namely \( g(x) = \sqrt{|x|} \)). But there is no function \( g : \mathbb{R} \rightarrow \{ x \in \mathbb{R} : x \geq 0 \} \) such that \((f \circ g)(x) = x\) for all \( x \).
We have established that if $f : S \to T$ has an inverse, then $f$ must be one-to-one and onto. The converse is true too, and we leave the details for you to verify. A function $f : S \to T$ that is one-to-one and onto (and therefore invertible) is sometimes called a set-theoretic isomorphism or a bijection.
**Example** The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 5$ is one-to-one and onto (you should check this). Therefore it is invertible. To find the inverse, we consider the equation

$$f \circ f^{-1}(x) = x$$

or

$$f(f^{-1}(x)) = x.$$
We may write this out as

$$3f^{-1}(x) + 5 = x.$$ 

Solving for $f^{-1}(x)$ gives

$$f^{-1}(x) = \frac{x - 5}{3}.$$ 

Check for yourself that $f \circ f^{-1}(x) = x$ and $f^{-1} \circ f(x) = x$. 

Example The function \( f(x) = x^3 + 1 \) is one-to-one and onto. Therefore it is invertible. To find the inverse, we consider the equation

\[
f \circ f^{-1}(x) = x
\]

or

\[
f(f^{-1}(x)) = x.
\]

We may write this out as

\[
[f^{-1}(x)]^3 + 1 = x.
\]

Solving for \( f^{-1}(x) \) gives

\[
f^{-1}(x) = \sqrt[3]{x - 1}.
\]
Example In this example we use some ideas from calculus.

The function \( f(x) = x^3 + x \) satisfies \( f'(x) = 3x^2 + 1 > 0 \) for every \( x \). Therefore \( f \) is strictly increasing. So it is one-to-one. Also \( f(x) \to +\infty \) as \( x \to +\infty \) and \( f(x) \to -\infty \) as \( x \to -\infty \). So \( f \) is onto. Therefore \( f \) is invertible.

It would be quite difficult to solve the equation

\[
f \circ f^{-1}(x) = x,
\]

and we shall not attempt to do so.
**Example**  The function $f : \mathbb{R} \to \mathbb{R}$ that is given by $f(x) = x^3$ is a bijection. You should check the details of this assertion for yourself. The inverse of this function $f$ is the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^{1/3}$. 
We leave it as an exercise for you to verify that the composition of two bijections (when the composition makes sense) is a bijection.