Math 310
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Figure: This is your instructor.

Cantor's Notion of Cardinality

One of the most profound ideas of modern mathematics is Georg Cantor's of the infinite (Georg Cantor, 1845–1918). Cantor's insight was that infinite sets can be compared by size, just as finite sets can. For instance, we think of the number 2 as *less* than the number 3; so a set with two elements is "smaller" than a set with three elements. We would like to have a similar notion of comparison for infinite sets. In this section we will present Cantor's ideas; we will also give precise definitions of the terms "finite" and "infinite."

Definition Let A and B be sets. We say that A and B have the *same cardinality* if there is a function f from A to B which is both one-to-one and onto (that is, f is a bijection from A to B). We write $\operatorname{card}(A) = \operatorname{card}(B)$.

Example Let $A = \{1, 2, 3, 4, 5\}$, $B = \{\alpha, \beta, \gamma, \delta, \varepsilon\}$, and $C = \{a, b, c, d, e, f\}$. Then A and B have the same cardinality because the function

$$f = \{(1, \alpha), (2, \beta), (3, \gamma), (4, \delta), (5, \varepsilon)\}$$

is a bijection of A to B. This function is not the *only* bijection of A to B (can you find another?), but we are only required to produce one.

On the other hand, A and C do not have the same cardinality; neither do B and C.

Georg Cantor (1845–1918)

Georg Waldemar Cantor was born in Denmark. Cantor attended primary school in St. Petersburg, then in 1856 the family moved to Germany. At first they lived in Wiesbaden, then they moved to Frankfurt. After attending the Höhere Gewerbeschule in Darmstadt, from 1860 he entered the Polytechnic of Zürich in 1862. Cantor attended lectures by Weierstrass, Kummer, and Kronecker. In 1869 he presented his thesis on number theory, and received his habilitation.

At Halle the direction of Cantor's research turned away from number theory and towards analysis. This was due to Heine, who challenged Cantor to establish the uniqueness of trigonometric series. Cantor solved this problem in April, 1870. Cantor was promoted to Extraordinary Professor at Halle in 1872 and in that year began a friendship with Dedekind.

In 1873 Cantor proved the rational numbers to be countable. He also showed that the algebraic numbers are countable. In 1874 he was able to prove that the real numbers are uncountable.

Liouville established in 1851 that transcendental numbers exist. Twenty years later, in his 1874 work, Cantor showed that, in a certain sense, "almost all" numbers are transcendental.

A major paper on dimension which Cantor submitted to Crelle's Journal in 1877 was treated with suspicion by Kronecker, and only published after Dedekind intervened. Cantor resented Kronecker's opposition to his work and never again submitted papers to Crelle's journal.

Almost at the same time as the Cantor-Dedekind correspondence ended, Cantor began another important correspondence with Mittag-Leffler. Soon Cantor was publishing in Mittag-Leffler's journal *Acta Mathematica*. At the end of May, 1884, Cantor had the first recorded attack of depression. He recovered after a few weeks but now seemed less confident.

Mathematical worries began to trouble Cantor around 1884; in particular her began to worry that he could not prove the continuum hypothesis.

In 1897, Cantor discovered the first of the paradoxes in set theory. Cantor began a correspondence with Dedekind to try to understand how to solve the problems but recurring bouts of his mental illness forced him to stop writing. Wheneer Cantor suffered from periods of depression he tended toward philosophy; he spent time trying to prove that Francis Bacon wrote the plays of Shakespeare.

A major event planned in Halle to mark Cantor's 70th birthday in 1915 had to be cancelled because of the war, but a smaller event was held in his home. In June, 1917 he entered a sanatorium for the last time and continually wrote to his wife asking to be allowed to go home. He died of a heart attack.

Notice that if $\text{card}(A) = \text{card}(B)$ via a function f_1 and $\text{card}(B) = \text{card}(C)$ via a function f_2, then $\text{card}(A) = \text{card}(C)$ via the function $f_2 \circ f_1$.

Definition Let A and B be sets. If there is a one-to-one function from A to B but no bijection between A and B, then we will write

$$\text{card}(A) < \text{card}(B)\,.$$

This notation is read "A has smaller cardinality than B."

We use the notation

$$\text{card}(A) \leq \text{card}(B)$$

to mean that either $\text{card}(A) < \text{card}(B)$ or $\text{card}(A) = \text{card}(B)$.

Notice that $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(B) \leq \text{card}(C)$ imply that $\text{card}(A) \leq \text{card}(C)$. Moreover, if $A \subset B$, then the inclusion map $i(a) = a$ is a one-to-one function of A into B; therefore $\text{card}(A) \leq \text{card}(B)$.

Example Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6, 8, 10\}$. Then the function

$$f(x) = 2x$$

is a one-to-one function from A to B. There is no one-to-one function from B to A (why not?). So we may write

$$\mathrm{card}(A) < \mathrm{card}(B)\,.$$

Now let S be the integers $\mathbb{Z}$ and let T be the rational numbers $\mathbb{Q}$. Certainly the function

$$h(x) = x$$

is one-to-one from S to T. It is not clear (but see below) whether there is a one-to-one function from T to S. We may in any event write

$$\mathrm{card}(S) \leq \mathrm{card}(T)\,.$$

The next theorem gives a useful method for comparing the cardinality of two sets.

Theorem [Schroeder–Bernstein] *Let A, B, be sets. If there is a one-to-one function $f : A \to B$ and a one-to-one function $g : B \to A$, then A and B have the same cardinality.*

Remark This remarkable theorem says that, if we can find an injection from A to B and an injection from B to A, then in fact there exists a single map that is a *bijection* of A to B. Observe that the two different injections may be completely unrelated. Often it is much easier to construct two separate injections than it is to construct a single bijection.

Proof: It is convenient to assume that A and B are disjoint; we may arrange this if necessary by replacing A by $\{(a, 0) : a \in A\}$ and B by $\{(b, 1) : b \in B\}$. Let D be the image of f and let C be the image of g. Let us define a *chain* to be a sequence of elements of either A or B—that is, a function $\phi : \mathbb{N} \to (A \cup B)$—such that

- $\phi(1) \in B \setminus D$;
- If for some j we have $\phi(j) \in B$, then $\phi(j + 1) = g(\phi(j))$;
- If for some j we have $\phi(j) \in A$, then $\phi(j + 1) = f(\phi(j))$.

We see that a chain is a sequence of elements of $A \cup B$ such that the first element is in $B \setminus D$, the second in A, the third in B, and so on. Obviously each element of $B \setminus D$ occurs as the first element of at least one chain.

Define $\mathcal{S} = \{a \in A : a \text{ is some term of some chain}\}$. It is helpful to note that

$$\begin{aligned}\mathcal{S} = \{x \in A : x &\text{ can be written in the form} \\ &x = g(f(g(\cdots f(g(y))\ldots))) \text{ for some } y \in B \setminus D\}. \quad (*)\end{aligned}$$

Observe that $\mathcal{S} \subset C$.

We set

$$k(x) = \begin{cases} f(x) & \text{if} \quad x \in A \setminus \mathcal{S} \\ g^{-1}(x) & \text{if} \quad x \in \mathcal{S} \end{cases}$$

Note that the second half of this definition makes sense because $\mathcal{S} \subset C$ and because g is one-to-one. Then $k : A \to B$. We shall show that in fact k is a bijection.

First notice that f and g^{-1} are one-to-one. This is not quite enough to show that k is one-to-one, but we now reason as follows: If $f(x_1) = g^{-1}(x_2)$ for some $x_1 \in A \setminus S$ and some $x_2 \in S$, then $x_2 = g(f(x_1))$. But, by $(*)$, the fact that $x_2 \in S$ now implies that $x_1 \in S$. That is a contradiction. Hence k is one-to-one.

It remains to show that k is onto. Fix $b \in B$. We seek an $x \in A$ such that $k(x) = b$.

Case A: If $g(b) \in \mathcal{S}$, then $k(g(b)) \equiv g^{-1}(g(b)) = b$ hence the x that we seek is $g(b)$.

Case B: If $g(b) \notin \mathcal{S}$, then we claim that there is an $x \in A$ such that $f(x) = b$. Assume this claim for the moment.

Now the x that we just found must lie in $A \setminus \mathcal{S}$. For if not then x would be in some chain. Then $f(x)$ and $g(f(x)) = g(b)$ would also lie in that chain. Hence $g(b) \in \mathcal{S}$, and that is a contradiction. But $x \in A \setminus \mathcal{S}$ tells us that $k(x) = f(x) = b$. That completes the proof that k is onto. Hence k is a bijection.

To prove the claim that we made in Case B, notice that if there is no $x \in A$ with $f(x) = b$ then $b \in B \setminus D$. Thus some chain would begin at b. So $g(b)$ would be a term of that chain. Hence $g(b) \in \mathcal{S}$ and that is a contradiction.

The proof of the Schroeder–Bernstein theorem is complete.

$\square$