Math 310
November 2, 2020 Lecture

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Figure: This is your instructor.

The Axiom of Choice

The Axiom of Choice, first enunciated by Zermelo (1871–1953), is one of the most subtle of the axioms of set theory, and it has profound and mysterious implications. The books [JEC], [RR1], and [[RR2] treat the Axiom of Choice in great detail.

It is not difficult to see that the well ordering of a set is closely related to the Axiom of Choice (see Section 4.2 for terminology). For if S is a set that is well ordered by a relation $\mathcal{R}$, then we may let $f : \mathcal{P}(S) \to S$ be the function that assigns to each $A \in \mathcal{P}(S)$ the (perforce unique) minimal element of A. The converse statement is true as well; but its proof involves ordinal arithmetic and transfinite mathematical induction and cannot be treated at this time.

Here is a way to specify a well ordering of any countable set, such as the rationals. The method seems artificial; but it *has* to be. The natural ordering on these sets will not do the job.

Let S be a countable set. Let $\phi : S \to \mathbb{N}$ be a set-theoretic isomorphism. If $x, y \in S$ are distinct elements, then we say that $(x, y) \in \mathcal{R}$ if $\phi(x) < \phi(y)$. Check for yourself that this creates a strict, simple order on S that well orders S.

As previously noted, it is impossible to give explicitly a well ordering of the real numbers; however, it is a theorem that *any* set can be well ordered (see [SUP]). The proof uses the Axiom of Choice.

In Section 4.5 we briefly mentioned the Continuum Hypothesis: that there are no cardinalities strictly between the cardinality of the integers and the cardinality of the continuum (the cardinality of the reals). Call this statement C. In 1938 Gödel showed that C could be added to the axioms of set theory and no contradiction would ensue. That is to say, there is a "model" for set theory in which the usual axioms of set theory are true and so is C. In 1963 Paul Cohen showed that instead $\sim C$ could be added to the axioms of set theory and no contradiction would ensue. That is, there is a model for set theory in which the usual axioms of set theory are true but C is false (i.e., $\sim C$ is true).

In logical terms, we say that the continuum hypothesis is *independent* of the other axioms of set theory. In particular, it is independent of the Axiom of Choice. The assertion C can never be proved as a theorem from the other axioms, nor can $\sim C$ be proved. Georg Cantor's inability to resolve the truth or falsity of C was a strong contributing factor to his debilitating mental illness at the end of his life. Sadly, mathematical logic was not sufficiently developed in Cantor's time for him to have been able to understand the ultimate resolution of the problem.

There are a number of very standard and useful tools in mathematics that are consequences of the Axiom of Choice. We now enunciate two of them.

In modern mathematics, especially in algebra, Zorn's lemma plays a central role. It is used to proved the existence of maximal ideals, of bases for vector spaces, and of other "maximal sets."

We need two pieces of terminology in order to formulate Zorn's lemma. First, if $(S, \leq)$ is a partially ordered set, then a *chain* in S is a subset $D \subset S$ that is linearly ordered (i.e., any two elements are comparable). An element u is an upper bound of the chain D if $d \leq u$ for every $d \in D$.

A typical enunciation of Zorn's lemma is this:

Let $(S, \leq)$ be a nonempty, partially ordered set with the property that every chain in S has an upper bound. Then S has a maximal element, i.e., an element m such that $s \leq m$ for every $s \in S$.

Zorn's lemma is equivalent to the Axiom of Choice.

Zorn's lemma is commonly applied in algebra to prove the existence of various objects. For example, it can be used to establish the existence of a basis for any vector space (see [HER] for a discussion of these ideas, and for the associated terminology).

In what follows, a vector space V is a set with notions of addition and scalar multiplication. A set $\mathcal{B}$ is a *basis* for V if any element $v \in V$ can be written as
$v = a_1 b_1 + a_2 b_2 + \cdots + a_k b_k$ for scalars a_j and elements $b_j \in \mathcal{B}$ and also the set $\mathcal{B}$ is minimal in size.

Proposition: *Let V be a vector space over the field F. Then V has a basis.*

Proof: Let $\mathcal{S}$ be the collection of all linearly independent sets in V. Partially order $\mathcal{S}$ as follows: if $A, B \in \mathcal{S}$, then $A \leq B$ if $A \subset B$. Now each chain clearly has a maximal element (or upper bound) since we may simply take the union of all the elements of the chain to be the upper bound. We then apply Zorn's lemma to conclude that the entire collection $\mathcal{S}$ has a maximal element. Call that maximal element Φ. We claim that Φ is a basis for V.

If the claim is not true, then there is an element x in V that is not in the span of Φ. But then x may be added to Φ, thereby contradicting the maximality of Φ. We conclude that Φ is the basis we seek. $\square$

Likewise, Zorn's lemma is used to establish the existence of maximal ideals in a ring, of algebraic closures, and of many other basic algebraic constructs. You will learn about these ideas in your course on abstract algebra.

The Hausdorff maximality principle is a variant of Zorn's lemma, also commonly used in algebraic applications.

Hausdorff's Maximality Principle: *If $\mathcal{R}$ is a transitive relation on a set S then there exists a maximal subset of S which is linearly ordered by $\mathcal{R}$.*

Hausdorff's principle is equivalent to the Axiom of Choice.

Just for fun, we conclude this section with a description of one of the most dramatic paradoxes in mathematics. This paradox stems from the Axiom of Choice, and is called the *Banach–Tarski paradox*. A version of this paradox is as follows:

Banach–Tarski Paradox [HJE], [JEC]: *It is possible to partition the solid ball in* $\mathbb{R}^3$*, of diameter one inch, into seven (disjoint) pieces in such a way that these seven pieces may be reassembled into a life-sized replica of the Statue of Liberty.*

We refer the reader to the Bibliography (in particular [JEC]) for a detailed discussion of this paradox. We should note that the paradox fails in dimension two; it holds in dimension three in part because of the complexity of the three dimensional rotation group (as contrasted with the rather simple two dimensional rotation group).

Of course the seven pieces into which we break the unit ball in the Banach–Tarski paradox are extremely pathological. The subject of measure theory was invented, in part, to rule out sets such as these. Measure theory is another subject, like axiomatic set theory, in which there are very specific rules limiting the ways in which sets may be created.