Math 310
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Figure: This is your instructor.

The Integers

Now we will apply the notion of an equivalence class to *construct* the integers (positive and negative and zero). There is an important point of knowledge to be noted here. In view of the last lecture, we may take the natural numbers as given. The natural numbers are universally accepted, and we have indicated how they may be constructed in a formal manner. However, the number zero and the negative numbers are a different matter. It was not until the fifteenth century that the concepts of zero and negative numbers started to take hold—for they do not correspond to explicit collections of objects (five fingers or ten shoes) but rather to *concepts* (zero books is the lack of books; minus 4 pens means that we owe someone four pens). After some practice we get used to negative numbers, but explaining in words what they mean is always a bit clumsy.

In fact, it is sobering to realize that the Italian mathematicians of the fifteenth and sixteenth centuries referred to negative numbers—in their *formal writings*—as "fictitious" or "absurd." Mathematics is, in part, a subject that we must get used to. It took several hundred years for mankind to get used to negative numbers.

It is much more satisfying, from the point of view of logic, to *construct* the integers from what we already have, that is, from the natural numbers. We proceed as follows. Let $A = \mathbb{N} \times \mathbb{N}$, the set of ordered pairs of natural numbers. We define a relation $\mathcal{R}$ on A as follows:

$$(a, b) \text{ is related to } (a^*, b^*) \text{ if } a + b^* = a^* + b$$

Theorem *The relation $\mathcal{R}$ is an equivalence relation.*

Proof: That (a, b) is related to (a, b) follows from the trivial identity $a + b = a + b$. Hence $\mathcal{R}$ is reflexive. Second, if (a, b) is related to (a^*, b^*), then $a + b^* = a^* + b$ hence $a^* + b = a + b^*$ (just reverse the equality) hence (a^*, b^*) is related to (a, b). So $\mathcal{R}$ is symmetric.

Finally, if (a, b) is related to (a^*, b^*) and (a^*, b^*) is related to (a^{**}, b^{**}), then we have

$$a + b^* = a^* + b \qquad \text{and} \qquad a^* + b^{**} = a^{**} + b^* .$$

Adding these equations gives

$$(a + b^*) + (a^* + b^{**}) = (a^* + b) + (a^{**} + b^*).$$

Cancelling a^* and b^* from each side finally yields

$$a + b^{**} = a^{**} + b.$$

Thus (a, b) is related to (a^{**}, b^{**}). Therefore $\mathcal{R}$ is transitive. We conclude that $\mathcal{R}$ is an equivalence relation. $\square$

Remark We cheated a bit in the proof of the last theorem. Since we do not yet have negative numbers, we therefore have not justified the process of "cancelling" that we used. The most rudimentary form of cancellation is Axiom **P4** of the natural numbers. Suggest a way to use mathematical induction, together with Axiom **P4**, to prove that if a, b, c are natural numbers and if $a + b = c + b$, then $a = c$.

Now our job is to understand the equivalence classes that are induced by $\mathcal{R}$. Let $(a, b) \in A = \mathbb{N} \times \mathbb{N}$, and let $[(a, b)]$ be the corresponding equivalence class. If $b > a$, then we will denote this equivalence class by the integer $b - a$. For instance, the equivalence class $[(2, 7)]$ will be denoted by 5. Notice that if $(a^*, b^*) \in [(a, b)]$, then $a + b^* = a^* + b$ hence $b^* - a^* = b - a$ *as long as* $b > a$. Therefore the numeral that we choose to represent our equivalence class is *independent of which element of the equivalence class is used to compute it.*

If $(a, b) \in A$ and $b = a$, then we let the symbol 0 denote the equivalence class $[(a, b)]$. Notice that if (a^*, b^*) is any other element of this particular $[(a, b)]$, then it must be that $a + b^* = a^* + b$ hence $b^* = a^*$; therefore this definition is unambiguous.

If $(a, b) \in A$ and $a > b$, then we will denote the equivalence class $[(a, b)]$ by the symbol $-(a - b)$. For instance, we will denote the equivalence class $[(7, 5)]$ by the symbol -2. Once again, if (a^*, b^*) is related to (a, b), then the equation $a + b^* = a^* + b$ guarantees that our choice of symbol to represent $[(a, b)]$ is unambiguous.

Thus we have given our equivalence classes names, and these names *look just like* the names that we give to integers: there are positive integers, and negative ones, and zero. But we want to see that these objects *behave* like integers. (As you read on, use the informal mnemonic that the equivalence class $[(a, b)]$ stands for the integer $b - a$.)

First, do these new objects that we have constructed *add* correctly? Well, let $A = [(a, b)]$ and $C = [(c, d)]$ be two equivalence classes. *Define* their sum to be $A + C = [(a + c, b + d)]$. We must check that this is unambiguous. If $(\widetilde{a}, \widetilde{b})$ is related to (a, b) and $(\widetilde{c}, \widetilde{d})$ is related to (c, d), then of course we know that

$$a + \widetilde{b} = \widetilde{a} + b$$

and

$$c + \widetilde{d} = \widetilde{c} + d.$$

Adding these two equations gives

$$(a+c)+(\widetilde{b}+\widetilde{d})=(\widetilde{a}+\widetilde{c})+(b+d)$$

hence $(a+c, b+d)$ is related to $(\widetilde{a}+\widetilde{c}, \widetilde{b}+\widetilde{d})$. Thus adding two of our equivalence classes gives another equivalence class, as it should. We say that addition of integers is *well defined*.

This point is so significant that it bears repeating. Each integer is an equivalence class—that is, a *set*. If we are going to add two integers m and n by choosing an element from the set m and another element from the set n, then the operation that we define had better be independent of the choice of elements. This is another way of saying that we want the sum of two equivalence classes to be another equivalence class. We call this the concept of "well definedness."

Example To add 5 and 3, we first note that 5 is the equivalence class $[(2,7)]$ and 3 is the equivalence class $[(2,5)]$. We add them componentwise and find that the sum is

$$[(2,7)] + [2,5)] = [(2+2,7+5)] = [(4,12)] .$$

Which equivalence class is this answer? Looking back at our prescription for giving names to the equivalence classes, we see that this is the equivalence class that we called $12-4$ or 8. So we have rediscovered the fact that $5+3=8$.

Now let us add 4 and -9. The first of these is the equivalence class $[(3,7)]$, and the second is the equivalence class $[(13,4])$. The sum is therefore $[(16,11)]$, and this is the equivalence class that we call $-(16-11)$ or -5. That is the answer that we would expect when we add 4 to -9.

Next, we add -12 and -5. Previous experience leads us to expect the answer to be -17. Now -12 is the equivalence class $[(19,7)]$, and -5 is the equivalence class $[(7,2)]$. The sum is $[(26,9)]$, which is the equivalence class that we call -17.

Finally, we can see in practice that our method of addition is unambiguous. Let us redo the second example using $[(6, 10)]$ as the equivalence class denoted by 4 and $[(15, 6)]$ as the equivalence class denoted by -9. Then the sum is $[(21, 16)]$, and this is still the equivalence class -5, as it should be.