The Rational Numbers

Math 310
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Figure: This is your instructor.

The Rational Numbers

In this section we use the integers, together with a construction using equivalence classes, to build the rational numbers. Let A be the set $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. In other words, A is the set of ordered pairs (a, b) of integers subject to the condition that $b \neq 0$. [*Think of this ordered pair as ultimately "representing" the fraction a/b.*] We definitely want it to be the case that certain ordered pairs represent the same number.

For instance,

$$\frac{1}{2} \text{ should be the same number as } \frac{3}{6}.$$

This motivates our equivalence relation. Declare (a, b) to be related to (a^*, b^*) if $a \cdot b^* = a^* \cdot b$. [*Here we are thinking that the fraction* a/b *should equal the fraction* a^*/b^* *precisely when* $a \cdot b^* = a^* \cdot b$.]

Is this an equivalence relation? Obviously the pair (a, b) is related to itself, since $a \cdot b = a \cdot b$. Also the relation is symmetric: if (a, b) and (a^*, b^*) are pairs and $a \cdot b^* = a^* \cdot b$, then $a^* \cdot b = a \cdot b^*$. Finally, if (a, b) is related to (a^*, b^*) and (a^*, b^*) is related to (a^{**}, b^{**}), then we have both

$$a \cdot b^* = a^* \cdot b \quad \text{and} \quad a^* \cdot b^{**} = a^{**} \cdot b^* . \qquad (\star)$$

Multiplying the left sides of these two equations together and the right sides together gives

$$(a \cdot b^*) \cdot (a^* \cdot b^{**}) = (a^* \cdot b) \cdot (a^{**} \cdot b^*) . \qquad (\star\star)$$

If $a^* = \mathbf{0}$, then it follows immediately from $(\star)$ that both a and a^{**} must be zero. So the three pairs $(a, b), (a^*, b^*)$, and (a^{**}, b^{**}) are equivalent, and there is nothing to prove. So we may assume that $a^* \neq \mathbf{0}$. We know *a priori* that $b^* \neq \mathbf{0}$; therefore we may cancel common terms in the equation $(\star\star)$ to obtain

$$a \cdot b^{**} = b \cdot a^{**}.$$

Thus (a, b) is related to (a^{**}, b^{**}), and our relation is transitive. [Exercise: explain why it is correct to "cancel common terms" in the last step.]

The resulting collection of equivalence classes will be called the set of *rational numbers*, and we shall denote this set with the symbol $\mathbb{Q}$.

Example: The equivalence class $[(4, 12)]$ contains all of the pairs $(4, 12), (1, 3)$, $(-2, -6)$. (Of course it contains infinitely many other pairs as well.) This equivalence class represents the fraction $4/12$ which we sometimes also write as $1/3$ or $(-2)/(-6)$.

If $[(a, b)]$ and $[(c, d)]$ are rational numbers then we define their *product* to be the rational number

$$[(a \cdot c, b \cdot d)].$$

This is well defined (unambiguous), for the following reason. Suppose that (a, b) is related to $(\widetilde{a}, \widetilde{b})$ and (c, d) is related to $(\widetilde{c}, \widetilde{d})$. We would like to know that

$[(a, b)] \cdot [(c, d)] = [(a \cdot c, b \cdot d)]$ is the same equivalence class as $[(\widetilde{a}, \widetilde{b})] \cdot [(\widetilde{c}, \widetilde{d})] = [(\widetilde{a} \cdot \widetilde{c}, \widetilde{b} \cdot \widetilde{d})]$. In other words, we need to know that

$$(a \cdot c) \cdot (\widetilde{b} \cdot \widetilde{d}) = (\widetilde{a} \cdot \widetilde{c}) \cdot (b \cdot d). \qquad (*)$$

But our hypothesis is that

$$a \cdot \widetilde{b} = \widetilde{a} \cdot b \quad \text{and} \quad c \cdot \widetilde{d} = \widetilde{c} \cdot d.$$

Multiplying together the left sides and the right sides, we obtain

$$(a \cdot \widetilde{b}) \cdot (c \cdot \widetilde{d}) = (\widetilde{a} \cdot b) \cdot (\widetilde{c} \cdot d).$$

Rearranging, we have

$$(a \cdot c) \cdot (\widetilde{b} \cdot \widetilde{d}) = (\widetilde{a} \cdot \widetilde{c}) \cdot (b \cdot d).$$

But this is just $(*)$. So multiplication is well defined.

Example: The product of the two rational numbers $[(3, 8)]$ and $[(-2, 5)]$ is

$$[(3 \cdot (-2), 8 \cdot 5)] = [(-6, 40)] = [(-3, 20)].$$

This is what we expect: the product of $3/8$ and $-2/5$ is $-3/20$.

If $q = [(a, b)]$ and $r = [(c, d)]$ are rational numbers and if r is not zero (that is, $[(c, d)]$ is not the equivalence class zero—in other words, $c \neq \mathbf{0}$), then we define the quotient q/r to be the equivalence class

$$[(ad, bc)].$$

We leave it to you to check that this operation is well defined.

Example: The quotient of the rational number $[(4, 7)]$ by the rational number $[(3, -2)]$ is, by definition, the rational number

$$[(4 \cdot (-2), 7 \cdot 3)] = [(-8, 21)].$$

This is what we expect: the quotient of $4/7$ by $-3/2$ is $-8/21$.

How should we add two rational numbers? We could try declaring $[(a,b)] + [(c,d)]$ to be $[(a+c, b+d)]$, but this will not work (think about the way that we usually add fractions). Instead we define

$$[(a,b)] + [(c,d)] = [(a \cdot d + b \cdot c, b \cdot d)].$$

That this definition is well defined (unambiguous) is left for the exercises. We turn instead to an example.

Example: The sum of the rational numbers $[(3, -14)]$ and $[(9, 4)]$ is given by

$$[(3 \cdot 4 + (-14) \cdot 9, (-14) \cdot 4)] = [(-114, -56)] = [(57, 28)].$$

This coincides with the usual way that we add fractions :

$$-\frac{3}{14} + \frac{9}{4} = \frac{57}{28}.$$

Notice that the equivalence class $[(\mathbf{0}, \mathbf{1})]$ is the rational number that we usually denote by $\mathbf{0}$. It is the additive identity, for if $[(a, b)]$ is another rational number, then

$$[(\mathbf{0}, \mathbf{1})] + [(a, b)] = [(\mathbf{0} \cdot b + \mathbf{1} \cdot a, \mathbf{1} \cdot b)] = [(a, b)].$$

A similar argument shows that $[(\mathbf{0}, \mathbf{1})]$ times any rational number $[(a, b)]$ gives $[(\mathbf{0}, b)]$ or $\mathbf{0}$. By the same token, the rational number $[(\mathbf{1}, \mathbf{1})]$ is the multiplicative identity. We leave the details for you.

Of course the concept of subtraction is really just a special case of addition (that is $\alpha - \beta$ is the same thing as $\alpha + (-\beta)$). So we shall say nothing further about subtraction.

In practice we will write rational numbers in the traditional fashion:

$$\frac{2}{5}\,, \ \frac{-19}{3}\,, \ \frac{22}{2}\,, \ \frac{24}{4}\,, \ \dots\,.$$

In mathematics it is generally not wise to write rational numbers in mixed form, such as $2\frac{3}{5}$, because the juxtaposition of two numbers could easily be mistaken for multiplication. Instead, we would write this quantity as the improper fraction $13/5$.