Math 310
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Figure: This is your instructor.

Introduction

Now that we are accustomed to the notion of equivalence classes, the construction of the integers and of the rational numbers seems fairly natural. In fact, equivalence classes provide a precise language for declaring certain objects to be equal (or for identifying certain objects). We can now use the integers and the rationals as we always have done, with the added confidence that they are not simply a useful notation but that they have been *constructed.*

We turn next to the real numbers. We saw in Section 2.3 that the rational number system is not closed under the operation of taking square roots, for example. We know from calculus that for many other purposes the rational numbers are inadequate. It is important to work in a number system that is closed with respect to all the operations we shall perform.

While the rationals are closed under the usual arithmetic operations, they are not closed under the operation of taking *limits*. For instance, the sequence of rational numbers $3, 3.1, 3.14, 3.141, 3.1415$, $3.14159, \ldots$ consists of terms that seem to be getting closer and closer together, *seem* to tend to some limit, and yet there is no rational number which will serve as a limit (of course it turns out that the limit is π—an "irrational" number).

We will now deal with the real number system, a system that contains all limits of sequences of rational numbers (as well as all limits of sequences of real numbers). In fact, our plan will be as follows. In this section we shall treat all the requisite properties of the reals. And we shall prove some significant theorems about the real number system. The actuall construction of the reals is a bit tricky, and we shall discuss that matter in an Appendix at the end.

Definition:
Let A be an ordered set and X a subsat of A. The set X called *bounded above* if there is an element $b \in A$ such that $x \leq b$ for all $x \in X$. We call the element b an *upper bound* for the set X.

Example:

Let $A = \mathbb{Q}$ with the usual ordering. The set $X = \{x \in \mathbb{Q} : 2 < x < 4\}$ is bounded above. For example, the number 15 is an upper bound for X. So are the numbers 12 and 4. It is interesting to observe that no element of this particular X can be an upper bound for X. The number 4 is a good candidate, but 4 is not an element of X. In fact if $b \in X$ then $(b + 4)/2 \in X$ and $b < (b + 4)/2$, so b could not be an upper bound for X.

It turns out that the most convenient way to formulate the notion that the real numbers have "no gaps" (i.e., that all sequences that seem to be converging actually have something to converge to) is in terms of upper bounds.

Definition:

Let A be an ordered set and X a subset of A. An element $b \in A$ is called a *least upper bound* (or *supremum*) for X if b is an upper bound for X and there is no upper bound b^* for X with $b^* < b$. We denote the supremum/(least upper bound) of X by $\sup X$ or $\operatorname{lub} X$.

By its very definition, if a least upper bound exists, then it is unique.

Example:

In the last example, we considered the set X of rational numbers strictly between 2 and 4. We observed there that 4 is the least upper bound for X. Note that this least upper bound is not an element of the set X.

The set $Y = \{y \in \mathbb{Z} : -9 \leq y \leq 7\}$ has least upper bound 7. In this case, the least upper bound *is* an element of the set Y.

Notice that we may define a lower bound for a subset of an ordered set in a fashion similar to that for an upper bound: $\ell \in A$ is a lower bound for $X \subset A$ if $\ell \leq x$ for all $x \in X$. A *greatest lower bound* (or *infimum*) for X is then defined to be a lower bound ℓ such that there is no lower bound ℓ^* with $\ell^* > \ell$. We denote the infimum/(greatest lower bound) of X by $\inf X$ or $\text{glb}\, X$.

Example:

The set X in the above Examples has lower bounds -20, $0, 1, 2$, for instance. The greatest lower bound is 2, which is *not* an element of the set.

The set Y in the last example has lower bounds $-53, -22, -10, -9$, to name just a few. The number -9 is the greatest lower bound. It *is* an element of Y.

Example:

Let $S = \mathbb{Z} \subset \mathbb{R}$. Then S does not have either an upper bound or a lower bound.

The purpose that the real numbers will serve for us is as follows: they will contain the rationals, they will still be an ordered field, and *every nonempty subset which has an upper bound will have a least upper bound*. We formulate this property as a theorem.

Theorem:

There exists an ordered field $\mathbb{R}$ that (i) contains $\mathbb{Q}$ and (ii) has the property that any nonempty subset of $\mathbb{R}$ which has an upper bound has a least upper bound.

We shall not prove this theorem right now. The proof is in an Appendix which will come later.

The last property described in this theorem is called the Least Upper Bound Property of the real numbers. As mentioned previously, this theorem will be proved in the optional next section. Now we begin to realize why it is so important to *construct* the number systems that we will use. We are endowing $\mathbb{R}$ with a great many properties. Why do we have any right to suppose that there exists a number system with all these properties? We must produce one!

Let us begin to explore the richness of the real numbers. The next theorem states a property that is certainly not shared by the rationals (see Section 2.3). It is fundamental in its importance.

Theorem:

Let x be a positive real number. Then there is a positive real number y such that $y^2 = y \cdot y = x$.

The proof of this theorem is serious business. It is a fairly tricky calculation. You will want to get out your pencil and verify the details yourself.

Proof: We will use throughout this proof the fact (see Part **6** of Theorem 6.3.9 in the text) that if $0 < a < b$, then $a^2 < b^2$. Let

$$S = \{s \in \mathbb{R} : s > 0 \text{ and } s^2 < x\}.$$

Then S is not empty since $x/2 \in S$ if $x < 2$ and $1 \in S$ otherwise. Also S is bounded above since $x + 1$ is an upper bound for S. By the theorem above, the set S has a least upper bound. Call it y. Obviously $0 < \min\{x/2, 1\} \le y$ hence y is positive. We claim that $y^2 = x$. To see this, we eliminate the other two possibilities.

If $y^2 < x$, then set $\varepsilon = (x - y^2)/[4(x+1)]$. Then $\varepsilon > 0$ and

$$\begin{aligned}
(y+\varepsilon)^2 &= y^2 + 2 \cdot y \cdot \varepsilon + \varepsilon^2 \\
&= y^2 + 2 \cdot y \cdot \frac{x - y^2}{4(x+1)} + \frac{x - y^2}{4(x+1)} \cdot \frac{x - y^2}{4(x+1)} \\
&< y^2 + 2 \cdot \frac{y}{x+1} \cdot \frac{x - y^2}{4} + \frac{x - y^2}{4} \cdot \frac{x}{4x} \\
&< y^2 + \frac{x - y^2}{2} + \frac{x - y^2}{16} \\
&< y^2 + (x - y^2) \\
&= x \,.
\end{aligned}$$

Thus $y + \varepsilon \in S$, and y cannot be an upper bound for S. This contradiction tells us that $y^2 \not< x$.

Similarly, if it were the case that $y^2 > x$, then we set $\varepsilon = (y^2 - x)/[4(x+1)]$. A calculatIon like the one we just did then shows that $(y-\varepsilon)^2 > x$. Hence $y - \varepsilon$ is also an upper bound for S, and y is therefore not the *least* upper bound. This contradiction shows that $y^2 \not> x$.

The only remaining possibility is that $y^2 = x$. That completes the proof. □

A similar proof shows that, if n is a positive integer and x is a positive real number, then there is a positive real number y such that $y^n = x$.

We next use the Least Upper Bound Property of the real numbers to establish two important qualitative properties of the real numbers:

Theorem:

The set $\mathbb{R}$ of real numbers satisfies the Archimedean Property:

Let a and b be positive real numbers. Then there is a natural number n such that $na > b$.

Proof: Suppose the Archimedean Property to be false. Then $S = \{na : n \in \mathbb{N}\}$ has b as an upper bound. Therefore S has a finite supremum β. Since $a > 0, \beta - a < \beta$. So $\beta - a$ is not an upper bound for S, and there must be a natural number n^* such that $n^* \cdot a > \beta - a$. But then $(n^* + 1)a > \beta$, and β cannot be the supremum for S. This contradiction proves the theorem. □