Math 310
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Figure: This is your instructor.

Key Properties of the Real Numbers

At the end of the last lesson we proved the Archimedean property of the real numbers. Now let us treat the closely related density property of the rationals. It says that between any two real numbers is a rational number. It is also the case that between any two rational numbers is an irrational number. Since one set is countable and the other uncountable, this is obviously a complicated situation.

Theorem:

The set $\mathbb{Q}$ of rational numbers satisfies the following Density Property:

> *Let $c < d$ be real numbers. Then there is a rational number q with $c < q < d$.*

Proof: Let $\lambda = d - c > 0$. By the Archimedean Property, choose a positive integer N such that $N \cdot \lambda > 1$. Again, the Archimedean Property gives a natural number P such that $P > |N \cdot c|$ and another Q such that $Q > |-N \cdot c|$. Then $Q > -N \cdot c$, and we see that Nc falls between the integers $-Q$ and P; therefore, there must be an integer M between $-Q$ and P (inclusive) such that

$$M - 1 \leq Nc < M.$$

Thus $c < M/N$. Also

$$M \leq Nc + 1 \text{ hence } \frac{M}{N} \leq c + \frac{1}{N} < c + \lambda = d.$$

So M/N is a rational number lying strictly between c and d. $\square$

One of the most profound and useful properties of the real numbers, and one that is equivalent to the Least Upper Bound Property, is the Intermediate Value Property:

Theorem:

Let f be a continuous, real-valued function with domain the interval $[a, b]$. If $f(a) = \alpha$, $f(b) = \beta$, and if $\alpha < \gamma < \beta$, then there is a value $c \in (a, b)$ such that $f(c) = \gamma$. Refer to the figure below.

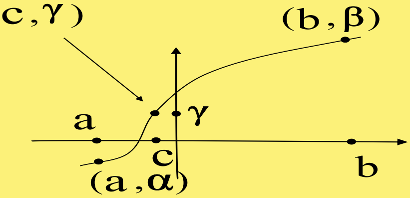

Figure: The Intermediate Value Theorem.

Proof:

Let

$$S = \{x \in [a, b] : f(x) < \gamma\} .$$

Then $S \neq \emptyset$ since $a \in S$. Moreover, S is bounded above by b. So $c = \sup S$ exists as a finite real number. We claim that $f(c) = \gamma$.

Clearly $f(c) \leq \gamma$ since c is the limit of numbers at which f takes values less than γ (we use the continuity of f here). Suppose, seeking a contradiction, that $f(c) < \gamma$. Let $\epsilon = \gamma - f(c)$. By the continuity of f, we may select $\delta > 0$ such that $|t - c| < \delta$ implies that $|f(t) - f(c)| < \epsilon/2$. But then, for $t \in (c - \delta, c + \delta)$, $f(t) < f(c) + \epsilon/2 < \gamma$. It follows that $(c - \delta, c + \delta) \subset S$, so c cannot be the supremum of S. That is a contradiction. Therefore, $f(c) = \gamma$. □

As an application, we prove the following special case of a theorem of Brouwer:

Theorem:

Let $f : [0,1] \to [0,1]$ be a continuous function. Then f has a fixed point, in the sense that there is a point $c \in [0,1]$ such that $f(c) = c$.

Proof:

Seeking a contradiction, we suppose not. Then, in particular, $f(0) > 0$ and $f(1) < 1$. Now set $g(x) = x - f(x)$. We see that $g(0) = 0 - f(0) < 0$ and $g(1) = 1 - f(1) > 0$. By the Intermediate Value Property, there must therefore be a point c between 0 and 1 such that $g(c) = 0$. But this says that $f(c) = c$, as required. □

We conclude by recalling the "absolute value" notation:

Definition:

Let x be a real number. We define

$$|x| = \begin{cases} x & \text{if} \quad x > \mathbf{0} \\ \mathbf{0} & \text{if} \quad x = \mathbf{0} \\ -x & \text{if} \quad x < \mathbf{0} \end{cases}$$

The absolute value of a real number x measures the distance of x to $\mathbf{0}$.

The most important property of the absolute value is the *triangle inequality*:

$$|x + y| \leq |x| + |y| \, .$$

Construction of the Real Number System

We spent some time constructing the natural numbers, the integers, and the rational numbers. This was rewarding, because then we were able to show that the natural numbers are well ordered, that proof by induction is valid, that multiplication of negative numbers is natural and makes sense, and many other interesting properties of number systems.

The real number system is certainly our richest number system so far, and the one with the most profound properties. It is essential that we see how it is constructed. This construction is a bit technical, but not difficult. You will find it rewarding to study.

There are several techniques for constructing the real number system $\mathbb{R}$ from the rational number system $\mathbb{Q}$. We use the method of Dedekind (Julius W. R. Dedekind, 1831–1916) cuts because it uses a minimum of new ideas and is fairly brief.

Keep in mind that, throughout this section, our universe is the system of rational numbers $\mathbb{Q}$. We are *constructing* the new number system $\mathbb{R}$.

Definition:

A *cut* is a subset $\mathcal{C}$ of $\mathbb{Q}$ with the following properties:

(1) $\mathcal{C} \neq \emptyset$

(2) If $s \in \mathcal{C}$ and $t < s$, then $t \in \mathcal{C}$

(3) If $s \in \mathcal{C}$, then there is a $u \in \mathcal{C}$ such that $u > s$

(4) There is a rational number x such that $c < x$ for all $c \in \mathcal{C}$

You should think of a cut $\mathcal{C}$ as the set of all rational numbers to the left of some point in the real line (that is, it is an open half-line of rational numbers—see Figure 6.2).

Figure: A cut.

For example, the set $\{x \in \mathbb{Q} : x^2 < 2\} \cup \{x \in \mathbb{Q} : x < 0\}$ is a cut. Roughly speaking, it is the set of rational numbers to the left of $\sqrt{2}$. [Take care to note that $\sqrt{2}$ does not exist as a rational number; so we are using a circuitous method to specify this set.] Since we have not constructed the real line yet, we cannot define this cut in that simple way; we have to make the construction more indirect. But if you consider the four properties of a cut, they describe a set that looks like a "rational left half-line."

Notice that if $\mathcal{C}$ is a cut and $s \notin \mathcal{C}$ then any rational $t > s$ is also not in $\mathcal{C}$. Also, if $r \in \mathcal{C}$ and $s \notin \mathcal{C}$, then it must be that $r < s$.

Definition:

If $\mathcal{C}$ and $\mathcal{D}$ are cuts, then we say that $\mathcal{C} < \mathcal{D}$ provided that $\mathcal{C}$ is a subset of $\mathcal{D}$ but $\mathcal{C} \neq \mathcal{D}$.

Check for yourself that "$<$" is a strict, simple ordering on the set of all cuts. We note that $\mathcal{C} = \mathcal{D}$ if and only if $\mathcal{C} \subset \mathcal{D}$ and $\mathcal{D} \subset \mathcal{C}$.

Now we introduce operations of addition and multiplication that will turn the set of all cuts into a field.

Definition:

If $\mathcal{C}$ and $\mathcal{D}$ are cuts, then we define

$$\mathcal{C} + \mathcal{D} = \{c + d : c \in \mathcal{C}, d \in \mathcal{D}\}.$$

We define the cut $\hat{\mathbf{0}}$ to be the set of all negative rationals.

The cut $\hat{\mathbf{0}}$ will play the role of the additive identity. We are now required to check that field axioms **A1–A5** hold.

For **A1**, we need to see that $\mathcal{C} + \mathcal{D}$ is a cut. Obviously $\mathcal{C} + \mathcal{D}$ is not empty. If s is an element of $\mathcal{C} + \mathcal{D}$ and t is a rational number less than s, write $s = c + d$, where $c \in \mathcal{C}$ and $d \in \mathcal{D}$. Then $t - c < s - c = d \in \mathcal{D}$ so $t - c \in \mathcal{D}$; and $c \in \mathcal{C}$. Hence $t = c + (t - c) \in \mathcal{C} + \mathcal{D}$. A similar argument shows that there is an $r > s$ such that $r \in \mathcal{C} + \mathcal{D}$. Finally, if x is a rational upper bound for $\mathcal{C}$ and y is a rational upper bound for $\mathcal{D}$, then $x + y$ is a rational upper bound for $\mathcal{C} + \mathcal{D}$. We conclude that $\mathcal{C} + \mathcal{D}$ is a cut.

Since addition of rational numbers is commutative, it follows immediately that addition of cuts is commutative. Associativity follows in a similar fashion. That takes care of **A2** and **A3**.

Now we show that if $\mathcal{C}$ is a cut, then $\mathcal{C} + \hat{\mathbf{0}} = \mathcal{C}$. For if $c \in \mathcal{C}$ and $z \in \hat{\mathbf{0}}$, then $c + z < c + \mathbf{0} = c$ hence $\mathcal{C} + \hat{\mathbf{0}} \subset \mathcal{C}$. Also, if $c^* \in \mathcal{C}$ then choose a $d^* \in \mathcal{C}$ such that $c^* < d^*$. Then $c^* - d^* < \mathbf{0}$, so $c^* - d^* \in \hat{\mathbf{0}}$. And $c^* = d^* + (c^* - d^*)$. Hence $\mathcal{C} \subset \mathcal{C} + \hat{\mathbf{0}}$. We conclude that $\mathcal{C} + \hat{\mathbf{0}} = \mathcal{C}$. This is **A4**.

Finally, for Axiom **A5**, we let $\mathcal{C}$ be a cut and set $-\mathcal{C}$ to be equal to $\{\mathbf{d} \in \mathbb{Q} : \exists \mathbf{d}^* > \mathbf{d} \text{ such that } \mathbf{c} + \mathbf{d}^* < \mathbf{0} \text{ for all } \mathbf{c} \in \mathcal{C}\}$. If x is a rational upper bound for $\mathcal{C}$, then $-x \in -\mathcal{C}$ so $-\mathcal{C}$ is not empty. It is also routine to check that $-\mathcal{C}$ is a cut. By its very definition, $\mathcal{C} + (-\mathcal{C}) \subset \hat{\mathbf{0}}$.

Further, if $z \in \hat{\mathbf{0}}$, then there is a $z^* \in \hat{\mathbf{0}}$ such that $z < z^*$. Choose an element $c \in \mathcal{C}$ such that $c + (z^* - z) \notin \mathcal{C}$ (why is this possible?). Let $c^* \in \mathcal{C}$ be such that $c < c^*$. Set $c^{**} = z - c^*$. Then $d^* = z - c > c^{**}$. We claim that $\widetilde{c} + d^* < \mathbf{0}$ for all $\widetilde{c} \in \mathcal{C}$. Suppose for the moment that this claim has been proved. Then this shows that $c^{**} \in -\mathcal{C}$. Then $z = c^* + c^{**} \in \mathcal{C} + (-\mathcal{C})$ so that $\hat{\mathbf{0}} \subset \mathcal{C} + (-\mathcal{C})$. We then conclude that $\mathcal{C} + (-\mathcal{C}) = \hat{\mathbf{0}}$, and Axiom **A5** is established.

It remains to prove the claim. So let d^* be defined as above, and select $\widetilde{c} \in \mathcal{C}$. Then

$$d^* + \widetilde{c} = z + (-c + \widetilde{c}) < z + (z^* - z) = z^* < \mathbf{0}.$$

Here we have used the choice of c. This establishes the claim and completes the proof of **A5**.

Having verified the axioms for addition, we turn now to multiplication.

Definition:

If $\mathcal{C}$ and $\mathcal{D}$ are cuts, then we define the product $\mathcal{C} \cdot \mathcal{D}$ as follows:

- If $\mathcal{C}, \mathcal{D} > \hat{0}$, then $\mathcal{C} \cdot \mathcal{D} = \{\mathrm{q} \in \mathbb{Q} : \mathrm{q} < \mathrm{c} \cdot \mathrm{d}\}$ for some $c \in \mathcal{C}$, $d \in \mathcal{D}$ with $c > 0, d > 0$.
- If $\mathcal{C} > \hat{0}, \mathcal{D} < \hat{0}$, then $\mathcal{C} \cdot \mathcal{D} = -(\mathcal{C} \cdot (-\mathcal{D}))$.
- If $\mathcal{C} < \hat{0}, \mathcal{D} > \hat{0}$, then $\mathcal{C} \cdot \mathcal{D} = -((-\mathcal{C}) \cdot \mathcal{D})$.
- If $\mathcal{C}, \mathcal{D} < \hat{0}$, then $\mathcal{C} \cdot \mathcal{D} = (-\mathcal{C}) \cdot (-\mathcal{D})$.
- If either $\mathcal{C} = \hat{0}$ or $\mathcal{D} = \hat{0}$, then $\mathcal{C} \cdot \mathcal{D} = \hat{0}$.

Notice that, for convenience, we have defined multiplication of negative numbers just as we did in high school. The reason is that the definition that we use for the product of two positive numbers cannot work when one of the two factors is negative (exercise).

We have said what the additive identity is in this realization of the real numbers. Of course the multiplicative identity is the cut corresponding to 1, or

$$\widehat{1} \equiv \{t \in \mathbb{Q} : t < 1\} .$$

We leave it to the reader to verify that if $\mathcal{C}$ is any cut, then $\widehat{1} \cdot \mathcal{C} = \mathcal{C} \cdot \widehat{1} = \mathcal{C}$.

Next time we shall complete the construction of the real number system using Dedekind cuts.