Math 310
December 7, 2020 Lecture

Steven G. Krantz

December 2, 2020

Figure: This is your instructor.

More on the Real Numbers

The real numbers are a profound and complex world. We earlier had an introduction to the real numbers, but we did not explore any of their truly deep properties.

In this current brief discussion we begin to explore the real numbers and establish some of their more remarkable aspects. This will be a real mathematical adventure, and you should prepare to enjoy it.

If x is a real number, then the *absolute value* of x, denoted $|x|$, is the distance of x to 0. In other words,

$$|x| = \begin{cases} x & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -x & \text{if} \quad x < 0 \,. \end{cases}$$

Fundamental to our study of the deeper properties of the real numbers is the *triangle inequality*: If x, y are real numbers, then

$$|x + y| \leq |x| + |y| \,. \qquad (*)$$

In fact the standard triangle inequality $(*)$ entails other inequalities that are also useful. Let $x = a + b$ and $y = -b$. Then $(*)$ implies

$$|(a + b) - b| \leq |a + b| + |b|$$

hence

$$|a| - |b| \leq |a + b| \,. \qquad (**)$$

A *sequence* in $\mathbb{R}$ is a function $\phi : \mathbb{N} \to \mathbb{R}$. We denote the elements of the sequence by $\phi(1), \phi(2), \ldots$. For example,

$$\phi(j) = j^2 + 1$$

is a sequence. It is often useful to write out the elements of the sequence in order: $2, 5, 10, 17, \ldots$. We frequently denote the elements of a sequence by the more convenient notation $\phi_1, \phi_2, \phi_3, \ldots$ (rather than think of the sequence as a function). The principal property of a sequence is whether or not it *converges*.

We say that a sequence $\{a_j\} = \{a_1, a_2, \ldots\}$ *converges* to a number α if, for every $\epsilon > 0$, there is a positive integer K such that $j > K$ implies that $|a_j - \alpha| < \epsilon$. What we have enunciated is a quantitative, rigorous way of asserting that the numbers a_j become closer and closer, and *stay* close, to α (within any desired distance ϵ).

Example:

Consider the sequence $\phi(j) = (-1)^j$, or

$$-1, 1, -1, 1, \ldots .$$

This sequence does *not* converge. Intuitively, the assertion is clear; because the numbers in the sequence do not get close and stay close to any fixed value α. To verify this claim rigorously, we suppose (seeking a contradiction) that in fact the sequence *does* converge to some number α. Let $\epsilon = 1/2$. Then, by the definition of convergence, there is a positive integer K such that if $j > K$, then $|\phi(j) - \alpha| < \epsilon = 1/2$. Choose $j > K$ so that $\phi(j) = 1$, that is to say, choose j even and greater than K. Then $\phi(j+1) = -1$.

As a result,

$$
\begin{aligned}
2 &= |1-(-1)| \\
&= |\phi(j)-\phi(j+1)| \\
&= \Big|(\phi(j)-\alpha)+(\alpha-\phi(j+1))\Big| \\
&\le |\phi(j)-\alpha|+|\alpha-\phi(j+1)| \\
&< \frac{1}{2}+\frac{1}{2}=1\,.
\end{aligned}
$$

We have derived the untenable assertion that $2<1$. This contradiction must mean that our assumption is false: the limit number α cannot exist. So the sequence has no limit.

Example:

Consider the sequence $\phi(j) = (-1)^j/j$. Intuitively, this sequence converges. For the elements of the sequence seem to be getting smaller and smaller in absolute value, and indeed seem to tend to zero. Let us prove that this actually is the case.

Let $\epsilon > 0$. There is a natural number K so large that $1/K < \epsilon$ (this is the Archimedian property of the natural numbers). If $j > K$, then

$$|\phi(j) - 0| = |\phi(j)| = \frac{1}{j} < \frac{1}{K} < \epsilon \,,$$

as was to be proved. So the sequence $\phi(j)$ converges to 0.

Let $\{a_j\}$ be a sequence. A *subsequence* of $\{a_j\}$ is a sequence $\{b_k\}$ whose elements come from the sequence $\{a_j\}$, in order. We usually denote the subsequence by $\{a_{j_k}\}$.

Example:

Let

$$a_1 = 1 \, , \; a_2 = 4 \, , \; a_3 = 9 \, , \; a_4 = 16 \, , \; a_5 = 25 \, , \; a_6 = 36 \, , \; \dots \, , \; a_j = j^2 \, , \; \dots$$

Then

$$a_{j_1} = 4 \, , \; a_{j_2} = 9 \, , \; a_{j_3} = 36 \, , \; a_{j_4} = 81 \, , \; \dots$$

is a subsequence. Of course a given sequence will have many different subsequences.

Some of the most commonly used subsets of the real numbers are intervals. The four types of intervals are these:

open	$(a, b) = \{x \in \mathbb{R} : a < x < b\} \quad .$
closed	$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\} \quad .$
half-open	$[a, b) = \{x \in \mathbb{R} : a \leq x < b\} \quad .$
half-open	$(a, b] = \{x \in \mathbb{R} : a < x \leq b\} \quad .$

A set $\mathcal{O} \subset \mathbb{R}$ is said to be *open* if, for any $x \in \mathcal{O}$, there is an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset \mathcal{O}$. A set $\mathcal{E} \subset \mathbb{R}$ is said to be *closed* if ${}^c\mathcal{E} \equiv \mathbb{R} \setminus \mathcal{E}$ is open.

A common mistake that students make is to supposed that if a set is not open then it is closed. Or if a set is not closed then it is open. This is incorrect. The set $[0, 1) = \{x \in \mathbb{R} : \mathbf{0} \leq x < 1\}$ is neither open nor closed.

Example:

Let $\mathcal{O} = \{x \in \mathbb{R} : x^2 < 1\}$. Then $\mathcal{O}$ is an open set. To see this, let $x \in \mathcal{O}$. Then certainly $|x| < 1$. Let $\epsilon = 1 - |x|$. Then we claim that $(x - \epsilon, x + \epsilon) \subset \mathcal{O}$. For if $t \in (x - \epsilon, x + \epsilon)$, then

$$|t| < |x| + |t - x| < |x| + \epsilon = |x| + (1 - |x|) = 1 \,.$$

Therefore $t^2 < 1$ and $t \in \mathcal{O}$. Thus $(x - \epsilon, x + \epsilon) \subset \mathcal{O}$. As a result, $\mathcal{O}$ is open.

Example:

Let $\mathcal{E} = \{x \in \mathbb{R} : x^2 \leq 1\}$. Then $\mathcal{E}$ is a closed set. To see this, we consider

$$^c\mathcal{E} = \{x \in \mathbb{R} : x < -1 \text{ or } x > 1\} \,.$$

Now let $x \in {}^c\mathcal{E}$. In case $x > 1$, then let $\epsilon = x - 1$. We claim that $(x - \epsilon, x + \epsilon) \subset {}^c\mathcal{E}$. For if $t \in (x - \epsilon, x + \epsilon)$, then

$$t \geq x - |x - t| > x - \epsilon = x - (x - 1) = 1 \,.$$

Thus $t \in {}^c\mathcal{E}$ so $(x - \epsilon, x + \epsilon) \subset {}^c\mathcal{E}$. A similar argument shows that in case $x < -1$ and $\epsilon = (-1) - x$, then $(x - \epsilon, x + \epsilon) \subset {}^c\mathcal{E}$. As a result, ${}^c\mathcal{E}$ is open; so $\mathcal{E}$ is closed.

Proposition: Let $\mathcal{E} \subset \mathbb{R}$ be a closed set. Let $\{a_j\}$ be a sequence of points in $\mathcal{E}$. If the sequence $\{a_j\}$ converges to a point $\alpha \in \mathbb{R}$, then $\alpha \in \mathcal{E}$. It is common to say that the set $\mathcal{E}$ contains all its limit points (or accumulation points).

Proof: Suppose that the assertion is false. Then there is a sequence $\{a_j\} \subset \mathcal{E}$ that converges to a point α, and $\alpha \in {}^c\mathcal{E}$. But $\mathcal{E}$ is closed ,so ${}^c\mathcal{E}$ is open. Therefore there is a number $\epsilon > \mathbf{0}$ such that $(\alpha - \epsilon, \alpha + \epsilon) \subset {}^c\mathcal{E}$. But then $a_j \notin (\alpha - \epsilon, \alpha + \epsilon)$ for every j. As a result, $|a_j - \alpha| \geq \epsilon$ for every j. Therefore it cannot be that $\{a_j\}$ converges to α, and that is a contradiction. $\square$