Solutions to HW 4

1. reflexive: If \( m \in \mathbb{Z} \), then \( m + m \) is even so \( m R m \).

symmetric: If \( m R n \), then \( m + n \) is even so \( n + m \) is even so \( n R m \).

transitive: If \( m R n \) and \( n R p \), then \( m + n \) is even and \( n + p \) is even. It follows that \((m + n) + (n + p) = (m + p) + 2n \) is even so \( m + p \) is even, hence \( m R p \). Thus this is an equivalence relation.

The equivalence classes are the even integers and the odd integers.

3. reflexive: If \((m, n) \in \mathbb{Z} \times \mathbb{Z}\), then \( m + n = m + n \), so \((m, n) R (m, n)\).

symmetric: If \((m, n) R (p, q)\), then \( m + q = n + p \), so \( p + n = n + q \). Hence \((p, q) R (m, n)\).

transitive: If \((m, n) R (p, q)\) and \((p, q) R (j, k)\), then \( m + q = n + p \) and \( p + k = q + j \). Adding left and right sides gives

\[ m + q + p + k = n + p + q + j \]

Cancelling \( p + q \) from both sides gives

\[ m + k = n + j \]

So \((m, n) R (j, k)\).

There is an equivalence class for each integer \( k \):

This is the set of \((m, n)\) such that \( m - n = k \).

A representative of this equivalence class is \((k, 0)\).
5. reflexive: If \((x, y) \in \mathbb{R}^2\), then \(x-x = 0\) is an integer and \(y-y = 0\) is an integer, so \((x, y) R (x, y)\).

Symmetric: If \((x, y) R (x', y')\), then \(x-x'\) is an integer and \(y-y'\) is an integer, so \(x'-x\) is an integer and \(y'-y\) is an integer. Hence \((x', y') R (x, y)\).

Transitive: If \((x, y) R (x, y')\) then \(x-x'\) is an integer and \(y-y'\) is an integer. If \((x, y') R (x, y'')\) then \(x'-x''\) is an integer and \(y'-y''\) is an integer. Then \(x-x'' = (x-x') + (x'-x'')\) is an integer and \(y-y'' = (y-y') + (y'-y'')\) is an integer, so \((x, y) R (x', y'')\).

An equivalence class is a lattice that forms the corners of squares of side 1:

![Diagram of a lattice with squares]

A representative of each equivalence class is the ordered pair \((x, y)\) with smallest \(|x| + |y|\). In the picture it would be the origin.
9. a) This is a function.
   b) This is a function.
   e) This is a function.

10. a) This is one-to-one because if
     \[ f(m) = f(n), \text{ then } m^2 + 2 = n^2 + 2 \text{ so } m = n. \]
     This is not onto because \( f(m) = 1 \).

   c) This is not one-to-one because \( h(0) = h(1) = 0 \).
   It is onto because \( x - x^3 = x \) can always be solved.

   e) This is one-to-one because it is strictly increasing.
   It is not onto because \( n \) such that \( g(n) = 1 \).

12. This is a strict single ordering. This is the way we order words in a dictionary: you first order by the first letter, then by the second letter, and so forth.

17. Define \( f(1) = 0, f(2) = 1, f(3) = -2, f(4) = 2, f(5) = -2 \), and so forth.
18. Define

\[ f(q) = \begin{cases} q & \text{if } q \in \mathbb{N} \\ 1 & \text{if } q \in \mathbb{Q} \setminus \mathbb{N} \end{cases} \]

20. a) \( \mathbb{N} \) is countable, \( \mathbb{Q} \) is countable, so \( \mathbb{N} \times \mathbb{Q} \) is countable.

b) \( \mathbb{N} \) is countable so \( \mathbb{N} \times \mathbb{N} \) is countable.

e) The function

\[ f : \mathbb{R} \to \mathbb{C} \]

\[ x \mapsto x + i0 \]

is one-to-one. So \( \text{card} (\mathbb{R}) \leq \text{card} (\mathbb{C}) \), hence \( \mathbb{C} \) is uncountable.

f) \( \mathbb{R} \) is uncountable, \( \mathbb{N} \) is countable. So \( \mathbb{R} \setminus \mathbb{N} \) is uncountable.

h) This contains all sequences of 3 and 7. So uncountable.

m) The set of all subsets of \( \mathbb{Z} \) is uncountable.
The set of all subsets of \( \mathbb{Z} \) with at most 6 elements is a subset of \( \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \). 
so is countable. Hence the set of all subsets of \( \mathbb{Z} \) with at least 6 elements is uncountable.

25.2) domain = \( \{ x \in \mathbb{R} : x \geq 0 \} \)

image = \( \{ y \in \mathbb{R} : y \geq -3 \} \)

b) domain = \( \{ x \in \mathbb{Q} : x \neq \pm 2 \} \)

image:
\[
y = \frac{1}{x^2 - y}
\]
\[
y \cdot (x^2 - y) = 1
\]
\[
x^2 - y = \frac{1}{y}
\]
\[
x^2 = y + \frac{1}{y}
\]

Clearly \( y \neq 0 \). Also \( y \) must be such that \( y + \frac{1}{y} \) has a real, non-zero square root. Finally, we must have \( y + \frac{1}{y} \geq 0 \).

e) domain = \( \{ x \in \mathbb{Q} : |x| < 1 \} \)

image = \( \{ y \in \mathbb{Q} : |y| < 1 \} \).