1. Let $0 < \epsilon < \frac{1}{3}$. Imagine constructing a Cantor-like set by first removing an open interval of length $\epsilon$, then removing two open intervals of length $\epsilon^2$, then removing four open intervals of length $\epsilon^3$, etc. At the $j$th step, we remove $2^{j-1}$ open intervals of length $\epsilon^j$.

The sum of the lengths of all the removed intervals (that is, the length of the complement of the Cantor set) is

$$\sum_{j=1}^{\infty} 2^{j-1} \epsilon^j = \sum_{j=0}^{\infty} 2^j \epsilon^{j+1} = \epsilon \sum_{j=0}^{\infty} (2\epsilon)^j = \frac{\epsilon}{1-2\epsilon}.$$ 

We will have $\frac{\epsilon}{1-2\epsilon} = \frac{\pi}{2}$ precisely when

$$\epsilon = \frac{\pi}{2} - 2\epsilon \pi$$

$$\epsilon \left( 1 + 2\pi \right)^2 = \frac{\pi}{2}$$

$$\epsilon = \frac{\pi}{1 + 2\pi}.$$ 

2. Choose $j$ so large that $3^{-j} < \frac{1}{2} |x_1 - x_2|$. Then the intervals that we remove at the $j$th step are shorter than $\frac{1}{2} |x_1 - x_2|$. So there will be no element of the complement of the Cantor set between $x_1$ and $x_2$. 

5. Let $x$ be an element of the interior $S'$ of $S$. Then there is an $\varepsilon > 0$ so that $(x-\varepsilon, x+\varepsilon) \subseteq S'$. Now if $t \in (x-\varepsilon, x+\varepsilon)$, let $s = \min \{ |t-(x-\varepsilon)|, |t-(x+\varepsilon)| \}$. Then $(t-s, t+s) \subset (x-\varepsilon, x+\varepsilon) \subseteq S$, so $t \in S'$. Hence $S'$ is open.

6. Denote the boundary of $S$ by $\partial S$. If $x \notin \partial S$ then there is an $\varepsilon > 0$ so that either $(x-\varepsilon, x+\varepsilon) \cap S = \emptyset$ or $(x-\varepsilon, x+\varepsilon) \cap S = \emptyset$. But then $(x-\varepsilon, x+\varepsilon) \subseteq \partial S$, so $\partial S$ is open. Hence $\partial S$ is closed.

9. If $q \in \emptyset$ and $\varepsilon > 0$ then $(q-\varepsilon, q+\varepsilon)$ will contain infinitely many rationals. So $\emptyset$ is not discrete. If $x \in C$, the Cantor set, then $x$ has an "address" which is a sequence of 0s and 1s. We may change the value of one of those digits arbitrarily far out in the address. That will produce an element of $C$ distinct from $x$ but arbitrarily close.
So \( C \) is not discrete.
If \( n \in \mathbb{Z} \), then \((n - \frac{1}{2}, n + \frac{1}{2}) \cap \mathbb{Z} = \{n\}\).
So \( \mathbb{Z} \) is discrete.

If
\[
\frac{1}{j} \in T,
\]
then \((\frac{1}{j - \frac{1}{(j+1)^2}}, \frac{1}{j + \frac{1}{(j+1)^2}}) \cap T = \{\frac{1}{j}\}\).
So \( T \) is discrete.

12. Let \( x \in \mathbb{R} \) and \( \varepsilon > 0 \). If \( x \) is irrational then choose \( k \) so large that \( \frac{1}{2^k} < \varepsilon \). Then \( x' = x + \frac{1}{2^k} \) is irrational and \((x - \varepsilon, x + \varepsilon)\) contains \( x' \).

If \( x \) is rational then choose \( k \) so large that \( \frac{\sqrt{2}}{2^k} < \varepsilon \). Then \( x' = x + \frac{\sqrt{2}}{2^k} \) is irrational and \((x - \varepsilon, x + \varepsilon)\) contains \( x' \).

Let \( \{\varrho_j\} \) be an enumeration of the rationals. Then this is a sequence that is dense in \( \mathbb{R} \).

16. Define a relation on \( \mathbb{U} \) by \( x \sim y \) if all points between \( x \) and \( y \) lie in \( \mathbb{U} \).
This is an equivalence relation. The set $U$ is the disjoint union of the equivalence classes, and each equivalence class is an open interval.

19. If $\{x_j\}$ is a enumeration of the rationals, then $\mathbb{R} = \mathbb{Q}$, so $\mathbb{R} \setminus \{x_j\}$ is the irrationals, which is uncountable.

If $\{x_j\}$ is the integers, then $\mathbb{Z} = \mathbb{Z}$ and $\mathbb{R} \setminus \{x_j\}$ is empty.

If $\{x_j\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$, then $\mathbb{R} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \cup \{0\}$, so $\mathbb{R} \setminus \{x_j\} = \{0\}$, a single point.

If $\{x_j\} = \{1, 2, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$, then $\mathbb{R} = \{1, 2, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \cup \{0, 2\}$, so $\mathbb{R} \setminus \{x_j\} = \{0, 2\}$ with two elements.

In a similar fashion we can arrange for $\mathbb{R} \setminus \{x_j\}$ to be a finite set of any size or a countable set.
22. Let \( x \in \mathbb{O} \), so \( x \in \mathbb{O}_j \) for some \( j \). Hence, there is a \( \varepsilon > 0 \) so that \( (x - \varepsilon, x + \varepsilon) \subset \mathbb{O}_j \subset \mathbb{O} \). So \( \mathbb{O} \) is open.

Let \( \mathbb{O}_j = (-\frac{1}{j}, 1 + \frac{1}{j}) \). Then
\[
\mathbb{O} = \bigcap_{j=2}^{\infty} \mathbb{O}_j = [0, 1] \text{ which is not open}
\]

Let \( E_j = [\frac{1}{j}, 1 - \frac{1}{j}] \). Then \( \bigcup_{j=2}^{\infty} E_j = (0, 1) \)
which is not closed.

If \( x \in \mathbb{O}_j \), then \( x \in E_j \) for some \( j \).
Since \( E_j \) is open, \( \varepsilon > 0 \) s.t. \( (x - \varepsilon, x + \varepsilon) \subset E_j \).
Hence \( (x - \varepsilon, x + \varepsilon) \subset \mathbb{O} \). So \( \mathbb{O} \) is open, and \( \mathbb{O} \) is closed.

26. We know that \( \mathbb{O} \) is the disjoint union of open intervals \( (a_j, b_j) \). Each such interval is the increasing union of closed intervals
\[
I_j^k = [a_j + \frac{k}{j}, b_j - \frac{k}{j}] \text{. Let}
\]
\[
E_k = \bigcup_{j=2}^{\infty} I_j^k
\]
Then $E_1 \subset E_2 \subset \ldots$ and $\cup E_k = \emptyset$
and each $E_k$ is closed,