SECOND MIDTERM

General Instructions: Read the statement of each problem carefully. Do only what is requested—nothing more and nothing less. Provide a complete solution to each problem.

If you only write the answer then you will not get full credit. If you need extra room for your work then use the backs of the pages.

Be sure to ask questions if anything is unclear.

(10 points) 1. Let $S$ be the set of all sequences of 0s and 1s. Prove, using the Cantor diagonalization argument, that $S$ is uncountable.
2. Let $S = \mathbb{N} \times \mathbb{N}$. If $\alpha = (a_1, a_2), \beta = (b_1, b_2) \in S$ then we say that $\alpha \sim \beta$ (that is, $\alpha$ is related to $\beta$) if

$$a_1 + b_2 = a_2 + b_1.$$  

(a) Prove that this is an equivalence relation.

(b) Let $\mathcal{S}$ denote the collection of equivalence classes. Define addition on $\mathcal{S}$ by

$$[(a_1, a_2)] + [(b_1, b_2)] = [(a_1 + b_1, a_2 + b_2)].$$

Prove that addition is well defined.
(3 points)  (c) What number system have you constructed in parts (a) and (b)?
    Explain.

3. Your universe is the set of rational numbers \( \mathbb{Q} \).

(3 points)  (a) What does it mean to say that a set has an \textit{upper bound}?

(3 points)  (b) What does it mean to say that a set has a \textit{least upper bound}?
(3 points)  (c) Give an example of a set of rationals that has an upper bound (in \( \mathbb{Q} \)) but not a least upper bound (in \( \mathbb{Q} \)).

(10 points)  4. Define the complex numbers—using the language of ordered pairs as we did in class. Define the arithmetic operations on \( \mathbb{C} \). Which ordered pair plays the role of \( i \)? What special property does \( i \) have? Prove it.
5. A function is a particular type of relation. State the two defining properties of a function.

6. Which of the following functions is one-to-one? Which is onto? Give explicit and detailed reasons for your answers.

(a) \( f : \mathbb{R} \to \mathbb{R} \), \( f(x) = x^2 \)

(b) \( g : \mathbb{R} \to \mathbb{R} \), \( g(x) = x^3 \)
(3 points) (c) $h : \mathbb{R} \to [-1, 1]$ , $h(x) = \sin x$

(3 points) (d) $m : \mathbb{R} \to (-\pi/2, \pi/2)$ , $m(x) = \tan^{-1} x$

7. State whether each of the following sets is countable or uncountable and give a specific reason for your answer.

(3 points) (a) $\mathbb{R} \times \mathbb{Z}$
(3 points) (b) \( \mathbb{Z} \cup (\mathbb{N} \times \mathbb{N}) \)

(3 points) (c) \((\mathbb{R} \times \mathbb{R}) \cap (\mathbb{C} \times \mathbb{Z})\)

(3 points) (d) \(\mathbb{C} \times \mathbb{C} \times \mathbb{N}\)
(10 points) 8. What does the Axiom of Choice say? Why is this axiom important?

(10 points) 9. Find all cube roots of the complex number $i + 1$. 
10. Explain Russell’s Paradox. That is, give the exact statement of the paradox and explain the logic behind it. How do we address the issues raised by Russell’s Paradox? What restrictions do we put on set theory as a result?