SOLUTIONS

Math 310
Krantz

SECOND MIDTERM EXAM

General Instructions: Read the statement of each problem carefully. If you want full credit on a problem then you must show your work. If you only write the answer then you will not receive full credit.

Be sure to ask questions if anything is unclear. This exam has 8 questions and is worth 100 points. You will have 50 minutes to take this exam.

(8 points) 1. Let \( S = \{2, 3, 4, 6\} \) and \( T = \{1, 3, 5, 7\} \). What is \( S \cup T \)? What is \( S \cap T \)?
What is \( S \setminus T \)?

\[
S \cup T = \{1, 2, 3, 4, 5, 6, 7\}
S \cap T = \{3\}
S \setminus T = \{2, 4, 6\}
\]
(8 points) 2. Let $S = \{2, 4, 6\}$ and $T = \{a, b, c, d\}$. What is $S \times T$? What is $T \times S$?

\[ S \times T = \{(2, a), (2, b), (2, c), (2, d), (4, a), (4, b), (4, c), (4, d),
(6, a), (6, b), (6, c), (6, d)\}\]

\[ T \times S = \{(a, 2), (a, 4), (a, 6), (b, 2), (b, 4), (b, 6), (c, 2), (c, 4),
(c, 6), (d, 2), (d, 4), (d, 6)\}\]

(8 points) 3. Draw a Venn diagram to illustrate the identity

\[ S \setminus (T \cup U) = (S \setminus T) \cap (S \setminus U) . \]
(6 points) 4. What is the power set of \( \{3, \alpha, x\} \)?

\[ \mathcal{P}(\{3, \alpha, x\}) = \{\emptyset, \{3\}, \{\alpha\}, \{x\}, \{3, \alpha\}, \{3, x\}, \{\alpha, x\}, \{3, \alpha, x\} \} \]

(12 points) 5. Which of these functions is one-to-one? Which is onto (give a brief reason for each answer)?

(a) \( f : \mathbb{R} \to \mathbb{R} \quad f(x) = x^3 + x \)

\[ f'(x) = 3x^2 + 1 > 0 \quad \text{f is increasing so one-to-one.} \]

\[ \lim_{x \to -\infty} f(x) = -\infty \quad \lim_{x \to \infty} f(x) = +\infty \quad \text{So f is onto.} \]

(b) \( g : \mathbb{N} \to \mathbb{N} \quad g(n) = n(n + 1) \)

\( g \) is increasing so one-to-one.

There is no \( n \) such that \( g(n) = 1 \), so not onto.

(c) \( h : \mathbb{R} \to \mathbb{R} \quad h(x) = x \sin x \)

\[ h(0) = h(\pi) = 0 \quad \text{so not one-to-one} \]

\( h \) is continuous and takes arbitrarily large positive and arbitrarily large negative values, so onto.
(12 points) 6. Which of these sets is countable and which uncountable (give a brief reason for each answer)?

(a) \( \mathbb{C} \times \mathbb{R} \)

\( p : \mathbb{R} \to \mathbb{C} \times \mathbb{R} \)

\( x \mapsto (0, x) \).

So \( \text{card}(\mathbb{C}) \leq \text{card}(\mathbb{C} \times \mathbb{R}) \)

hence uncountable

(b) \( \mathbb{Z} \times \mathbb{N} \)

The product of two countable sets is countable, so countable.

(c) \( \mathbb{Z} \times \mathbb{C} \)

\( h : \mathbb{C} \to \mathbb{Z} \times \mathbb{C} \)

\( z \mapsto (0, z) \).

So \( \text{card}(\mathbb{C}) \leq \text{card}(\mathbb{Z} \times \mathbb{C}) \)

hence uncountable.

(10 points) 7. Calculate the inverse of the function \( f : \mathbb{R} \to \mathbb{R} \) given by

\[
f(x) = \begin{cases} 
x & \text{if } x \leq 0 \\
x^2 & \text{if } x > 0. 
\end{cases}
\]

\[
f^{-1}(x) = \begin{cases} 
x & \text{if } x \leq 0 \\
\sqrt{x} & \text{if } x > 0. 
\end{cases}
\]
(6 points) 8. Prove that the collection of $S$ of irrational numbers is uncountable.

If the set of irrational numbers were countable then

$$IR = \mathbb{Q} \cup \{\text{irrationals}\}$$

would be countable. Contradiction.

So the set of irrationals is uncountable.
(8 points) 9. Explain why the product of a countable set and an uncountable set is uncountable.

Let $S$ be countable and $T$ uncountable.

Let $s_0 \in S$. Define $f: T \rightarrow S \times T$

$$t \rightarrow (s_0, t)$$

Then $f$ is one-to-one. So

$$\text{card}(T) \leq \text{card}(S \times T).$$

Since $T$ is uncountable, we conclude that

$S \times T$ is uncountable.

(6 points) 10. Explain why the union of a countable set and an uncountable set is uncountable.

Let $S$ be countable and $T$ be uncountable.

Then $T \subset S \cup T$. So

$$\text{card}(T) \leq \text{card}(S \cup T).$$

Since $T$ is uncountable, we conclude that

$S \cup T$ is uncountable.
(8 points) 11. Prove that addition in the integers is well defined. You should use the actual, rigorous definition of the integers (in terms of ordered pairs of natural numbers) to do this problem.

Let \((a, b) \sim (a', b')\) and \((c, d) \sim (c', d')\). We need to see that \((a+c, b+d) \sim (a'+c', b'+d')\). We know that \(a+b' = a'+b\) and \(c+d' = c'+d\). Adding left sides and right sides gives

\[a+b'+c+d' = a'+b+c'+d\]

or \((a+c)+(b+d) = (a'+c')+(b'+d')\).

So \((a+c, b+d) \sim (a'+c', b'+d')\) as desired.

(8 points) 12. What is the multiplicative inverse of the complex number \(2 - 3i\)?

\[
\frac{2 + 3i}{2^2 + 3^2} = \frac{2}{13} + \frac{3}{13}i
\]
(8 points) 13. Find a square root in the quaternions of the quaternion \(4 \cdot 1 + 4 \cdot k\).

**Guess a square root of the form** \(\alpha \cdot \frac{1}{2} + \beta \cdot \frac{k}{2}\).

So, \((\alpha \cdot \frac{1}{2} + \beta \cdot \frac{k}{2}) \cdot (\alpha \cdot \frac{1}{2} + \beta \cdot \frac{k}{2}) = 4 \cdot \frac{1}{2} + 4 \cdot \frac{k}{2}\).

\((\alpha^2 - \beta^2) \cdot \frac{1}{2} + 2 \alpha \beta \cdot \frac{k}{2} = 4 \cdot \frac{1}{2} + 4 \cdot \frac{k}{2}\)

\(\alpha^2 - \beta^2 = 4, \quad 2 \alpha \beta = 4 \Rightarrow \alpha = \frac{2}{\beta}\).

\((\frac{2}{\beta})^2 - \beta^2 = 4 \Rightarrow 4 - \beta^4 = 4 \beta^2 \Rightarrow \beta^4 + 4 \beta^2 - 4 = 0\)

\(\beta^2 = \frac{-4 \pm \sqrt{16 + 16}}{2} = -4 \pm 4 \sqrt{2} = -2 \pm 2 \sqrt{2}\).

\(\beta = \sqrt{2}, \alpha = \frac{2}{\sqrt{2}} = \sqrt{2}\).

**Solution** \(\frac{2}{\sqrt{2}} \cdot \frac{1}{2} + \sqrt{2} \cdot \frac{k}{2}\).

(6 points) 14. Find all cube roots of the complex number \(i\).

\(i = e^{\frac{i \pi}{2}}\)

Solve \(r^3 e^{\frac{3i \theta}{2}} = 1 \cdot e^{\frac{i \pi}{2}} \Rightarrow r = 1, \theta = \frac{\pi}{6}\).

\(z_1 = 1 \cdot e^{\frac{i \pi}{6}}\)

Solve \(r^3 e^{\frac{3i \theta}{2}} = 1 \cdot e^{\frac{i 5\pi}{2}} \Rightarrow r = 1, \theta = \frac{5\pi}{6}\).

\(z_2 = 1 \cdot e^{\frac{i 5\pi}{6}}\)

Solve \(r^3 e^{\frac{3i \theta}{2}} = 1 \cdot e^{\frac{i 9\pi}{2}} \Rightarrow r = 1, \theta = \frac{9\pi}{6} = \frac{3\pi}{2}\).

\(z_3 = 1 \cdot e^{\frac{i 3\pi}{2}}\).