THIRD MIDTERM

General Instructions: Read the statement of each problem carefully. Do only what is requested—nothing more and nothing less. Provide a complete solution to each problem. If you only write the answer then you will not get full credit. If you need extra room for your work then use the backs of the pages.

Be sure to ask questions if anything is unclear. This exam is worth 100 points.

(10 points) 1. Give an example of a subset $S$ of $\mathbb{R}$ that is both open and closed. Explain why it is both open and closed.

Let $S = [0, 1]$. Then $S$ is open because each $x \in S$ has the neighborhood $(x-\epsilon, x+\epsilon)$ that lies in $S$.

Also $S$ is closed because its complement $\mathbb{R} \setminus S$ is open.
2. Which of the following subsets of the reals are open, which are closed, and which are compact?

(2 points) (a) \( \{x \in \mathbb{R} : 0 \leq x < \infty\} \) Not open because it contains left endpoint. Closed because complement is open. Not compact because unbounded.

(2 points) (b) \( \{x \in \mathbb{R} : 0 < x < \infty\} \) Open. Each \( x \) in the set has a neighborhood \( (x-\varepsilon, x+\varepsilon) \) that lies in the set with \( \varepsilon = 1/x/2 \). Not closed because missing left endpoint. Not compact because unbounded.

(2 points) (c) \([0, 1)\) Not closed because missing right endpoint. Not open because contains left endpoint. Not compact because not closed.

(2 points) (d) \(\{1, 1/2, 1/3, \ldots\} \) Not open since contains no intervals. Not closed because does not contain limit point 0. Not compact because not closed.

(2 points) (e) \((0, 1)\) Open because \( x \) in the set has the neighborhood \( (x-\varepsilon, x+\varepsilon) \) in the set, where \( \varepsilon = \min\{1/x, 1-1/x\} \). Not closed because missing endpoints. Not compact because not closed.
(10 points) 3. Let $S$ be a subset of $\mathbb{R}$. Define the interior of $S$.

A point $x \in S$ is in the interior of $S$ if there is a neighborhood $(x-\varepsilon, x+\varepsilon)$ that lies in $S$.

(10 points) 4. Let $S$ be a subset of $\mathbb{R}$. Define the boundary of $S$.

A point $x$ is in the boundary of $S$ if each neighborhood $(x-\varepsilon, x+\varepsilon)$ of $x$ contains points of $S$ and points of $\mathbb{R} \setminus S$.

(10 points) 5. Prove that the collection of all endpoints of the removed intervals in the construction of the Cantor set is countable.

There are countably many removed intervals and each of these has 2 end points. Thus there are countably many end points.
6. **TRUE or FALSE:** The countable union of open sets is open.

   **TRUE.** Let \( \mathcal{O}_j \) be open for each \( j \). Let \( x \in \bigcup_{j=1}^{\infty} \mathcal{O}_j \).
   Then \( x \in \mathcal{O}_k \) for some \( k \). Hence \( \exists \, \varepsilon > 0 \) s.t.
   \((x-\varepsilon, x+\varepsilon) \subseteq \bigcup_{j=1}^{\infty} \mathcal{O}_j \).
   So \( \bigcup_{j=1}^{\infty} \mathcal{O}_j \) is open.

7. **TRUE or FALSE:** The countable union of closed sets is closed.

   **FALSE.** Let \( E_j = [\frac{1}{j}, 1-\frac{1}{j}] \). Then each \( E_j \) is closed but
   \[ \bigcup_{j=1}^{\infty} E_j = (0, 1) \]
   is open and not closed.
(10 points) 8. Write the set \((0, 1)\) as the countable increasing union of compact sets.

\[
(0, 1) = \bigcup_{j=1}^{\infty} \left[ \frac{1}{j}, 1 - \frac{1}{j} \right].
\]

(10 points) 9. Write the set \([0, 1]\) as the countable decreasing intersection of open sets.

\[
[0, 1] = \bigcap_{j=1}^{\infty} \left( -\frac{1}{j}, 1 + \frac{1}{j} \right).
\]
(10 points) 10. Explain why the Cantor set is uncountable.

We may assign an "address" to each element of the Cantor set as follows:

If \( x \) is in the left-hand interval of \( I_1 \), then the first entry in the address is 0.

If \( x \) is in the right-hand interval of \( I_1 \), then the first entry in the address is 1.

Inside the just indicated interval, if \( x \) is in the left interval of \( I_2 \), the second entry in the address is 0.

Otherwise it is 1.

Continue in this fashion.

So we assign to each element of the Cantor set a sequence of 0s and 1s, conversely, to each sequence of 0s and 1s there corresponds an element of the Cantor set. Thus the Cantor set is uncountable.