Section 6.1

1. Let $\varepsilon > 0$. Set $\delta = \varepsilon / c$. Then, for $\|x - y\| < \delta$,

\[ \|f(x) - f(y)\| \leq c \|x - y\| \leq c \delta = \varepsilon, \]

so $f$ is continuous.

Fix $x, x'$ are both fixed points then

\[ \|x - x'\| = \|f(x) - f(x')\| \leq c \|x - x'\|. \]

Since $c < 1$, this is false unless $\|x - x'\| = 0$ so $x = x'$.

2. Let $x = \sqrt{x^2 + 1}$. Then $x^2 + 1 = x^2$ so $L = 0$. Contradiction.

While $|f'(x)| < 1$ for all $x$, it is also true that

\[ f'(x) = \frac{x}{\sqrt{x^2 + 1}} \]

which is always close to 1 as $x \to \infty$, so there will be no $c < 1$. 
6.2) Define
\[ J = \sum_{k=0}^{\infty} H^k = I + H + H^2 + \cdots \]

Since \[ \| H \| < 1 \], we see that
\[ \| H^k \| \leq \| H \| \| H \|^{k-1} = \| H \|^k \]
and, in general,
\[ \| H^k \| \leq \| H \|^k \]
So \[ \sum H^k \] converges in norm. Moreover,
\[ H \cdot J = \sum_{k=0}^{\infty} H^{k+1} \]

So \[ (I - H) \cdot J = \sum_{k=0}^{\infty} H^k - \sum_{k=0}^{\infty} H^{k+1} = I \].

9.6) \[ g(x) = x^3 - 2 \], \[ x_0 = 5/4 \]
\[ x_1 = x_0 - \frac{g(x_0)}{g'(x_0)} = \frac{5}{4} - \frac{125/64 - 2}{3 \cdot 25/64} = \frac{5}{4} - \frac{5}{12} + \frac{32}{75} \]
\[ = \frac{5}{6} + \frac{32}{75} = \frac{63}{50} = 1.26 \]
\[ x_2 = x_1 - \frac{g(x_1)}{g'(x_1)} = \frac{63/50}{5/6} - \frac{(63/50)^3 - 2}{3(63/50)^2} \]
\[ = \frac{63}{50} - \frac{2(0.0009 - 2)}{1.76} = 1.26 - .000084 = 1.2599 \]
Note that $|g'| = |3x^2| \geq 3$ on $[1, 1.5]$
and $|g''| = |16x^3| \leq 9$ on $[2, 1.5]$

If we work in a interval on which $|g| \leq \frac{3}{9} - 1$
then we will converge to a root.
Such a interval is $[1, 1.3]$. 

Section 6.2

1, b) 

\[
J = \int f = \left( \frac{y_1 - x^2}{(x^2 + y^2)^2} \right) \cdot \left( \frac{y_2 - x^2}{(x^2 + y^2)^2} \right) \\
\]

\[
\det J = \left( \frac{y_1 - x^2}{(x^2 + y^2)^2} \right) \left( \frac{y_2 - x^2}{(x^2 + y^2)^2} \right) - \frac{4x^2 y_2^2}{(x^2 + y^2)^4} \\
= -x^4 - y_1 + 2x^2 y_2^2 - 4x^2 y_2^2 \\
\]

\[
= -\frac{(x^2 + y_2^2)^2}{(x^2 + y_2^2)^4} \\
= -\frac{1}{(x^2 + y_2^2)^2} \\
\]

This is non-vanishing at every point but $(0, 0)$,
So there will be a local inverse near every point
but $(0, 0)$. 

d) \( \det J = \begin{pmatrix} 1 & e^y \\ e^x & t \end{pmatrix} \)

\[ \det J = (1 - e^{x+y}). \]

This is nonzero, hence gives rise to a local inverse, except at point where \( x = -y \).

2. \( \alpha = u + v \)
\( \beta = uv \)

Then \( u = \beta \) so \( \alpha = \frac{\beta}{v} + v \)

\[ \alpha v = \beta + v^2 \]
\[ v^2 - \alpha v + \beta = 0 \]
\[ v = \frac{\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2} \]

Since \( 0 < v < u \), it is clear that \( \alpha^2 - 4\beta > 0 \).

For such \( \alpha, \beta \) we can take

\[ u = \frac{\alpha + \sqrt{\alpha^2 - 4\beta}}{2} \]
\[ v = \frac{\alpha - \sqrt{\alpha^2 - 4\beta}}{2} \]

That is \( \varphi(x, \beta) = \begin{pmatrix} \frac{x - \alpha^2 - 4\beta}{2} \\ \frac{x + \sqrt{\alpha^2 - 4\beta}}{2} \end{pmatrix} \) for example.
b) \( D_g = \left( \frac{1}{2} - \frac{\alpha}{2\sqrt{\alpha^2 - \eta \beta}} \right) \left( \frac{1}{\sqrt{\alpha^2 - \eta \beta}} \right) \) (1)

On the other hand,

\[ D_f = \begin{pmatrix} 1 & 1 \\ \sqrt{u} & u \end{pmatrix} \]

\[ D_f^{-1} = \begin{pmatrix} \frac{u}{u-v} & \frac{-1}{u-v} \\ \frac{-\sqrt{u}}{u-v} & \frac{1}{u-v} \end{pmatrix} \] (2)

If you plug (1) and (2) into (2),

then you will get (1).
3. b) \( D^F = \frac{x_1 e^{x_1 y} + 2y \cos x_1 x_2}{y} \)

At \((1, 0, 0)\), this equals \(1 - e^0 + 2 \cdot 0 \cdot \cos 0 = 1\).

It is nonvanishing, so can solve for \(y\) in terms of \(x_1\).

Note that \(\frac{D^F}{\partial x_1} = (y e^{x_1 y} - x_2 y^2 \cos x_1 x_2, x_1 e^{x_1 y} - x_1 y^2 \cos x_1 x_2)\).

\(D\phi(x) = -\left(\frac{D^F}{\partial y}\right) (\frac{\phi(x)}{\partial x})^{-1} \frac{D^F}{\partial x} (\phi(x))\).

1) \(\frac{D^F}{\partial y} = \begin{pmatrix} -2y_1 & -2y_2 \\ -1 & 1 \end{pmatrix}\).

At \((2, 1, 1)\), this equals \(\begin{pmatrix} -2 & -2 \\ -1 & 1 \end{pmatrix}\).

This matrix has nonzero determinant so is certainly invertible, hence can solve for \(y\) in terms of \(x_1\).
\[
\frac{\partial F}{\partial x} = (2x - 1).
\]

\[
D\phi(x) = -\left(\frac{\partial^2 F}{\partial x^2}(\phi(x))\right)^{-1} \frac{\partial^2 F}{\partial x \partial \phi(x)}(\phi(x)).
\]