Figure: This is your instructor.
2.2 Subsequences

Homework: 1, 2, 4, 8.

Given a sequence \( \{a_n\} \). For strictly increasing positive integers \( n_1 < n_2 < n_3 < \cdots \), the sequence 
\[
a_{n_1}, a_{n_2}, a_{n_3}, \ldots
\]
is called a subsequence of the sequence \( \{a_n\} \), denoted \( \{a_{n_k}\}_{k=1}^\infty \) or \( \{a_{n_k}\} \). So a subsequence contains a subcollection of terms of the original sequence in the same order.

Caution: \( a_{n_1}, a_{n_1}, a_{n_1}, \ldots \) cannot be called a subsequence of \( \{a_n\} \) as the subscripts (or indices) are not strictly increasing.

Example

Consider the sequence \( \{2^n\} = \{2, 4, 8, 16, \ldots\} \). Then the sequence \( \{2^{2k}\} = \{4, 16, 64, \ldots\} \) is a subsequence, with \( n_k = 2k \).

However, \( \{2^{1+(k-9)^2}\} \) is not a subsequence of \( \{2^n\} \) as the positive integers \( 1 + (k - 9)^2 \) are not strictly increasing as \( k \) increases.
Proposition

If \( \{a_j\} \) is a convergent sequence with limit \( \alpha \), then every subsequence of \( \{a_j\} \) converges to the limit \( \alpha \).

Conversely, if a sequence \( \{b_j\} \) has the property that each of its subsequences is convergent then \( \{b_j\} \) itself is convergent.

Proof. Assume that \( \{a_j\} \) converges to \( \alpha \), and let \( \{a_{j_k}\} \) be a subsequence. Let \( \epsilon > 0 \) and choose \( N > 0 \) such that \( |a_j - \alpha| < \epsilon \) whenever \( j > N \). Now if \( k > N \) then \( j_k > N \) hence \( |a_{j_k} - \alpha| < \epsilon \). Therefore, by definition, the subsequence \( \{a_{j_k}\} \) also converges to \( \alpha \).

The converse is trivial, simply because the sequence is a subsequence of itself.
Every real sequence has a monotone subsequence

The second part of the above proposition may be strengthened as follows: If every subsequence of \( \{b_j\} \) converges, then all the subsequences of \( \{b_j\} \) converge to the same limit. The proof of this claim is left as an exercise.

We now show a very interesting property of sequences of real numbers (called real sequences for short). We say a term \( a_j \) of a real sequence \( \{a_n\} \) is (forward) dominating if \( a_j \geq a_k \) for all \( k \geq j \), that is, \( a_j \) is greater than or equal to every term after it.

**Theorem**

Every real sequence \( (a_n) \) has a monotone subsequence.

**Proof.** If \( \{a_n\} \) contains infinitely many dominating terms \( a_{n_1}, a_{n_2}, \ldots \) with \( n_1 < n_2, \ldots \), then \( \{a_{n_k}\} \) is clearly a decreasing subsequence of \( \{a_n\} \).
Every real sequence has a monotone subsequence

Assume that \( \{a_n\} \) has *only finitely many* dominating terms. Let \( n_1 \) be a positive integer such that for every \( n \geq n_1 \), \( a_n \) is not dominating. Since \( a_{n_1} \) is not dominating, there is an \( n_2 \in \mathbb{N} \) such that \( n_1 < n_2 \) and \( a_{n_1} < a_{n_2} \). Since \( a_{n_2} \) is not dominating, there is an \( n_3 \in \mathbb{N} \) such that \( n_2 < n_3 \) and \( a_{n_2} < a_{n_3} \). Continuing this way, we obtain an increasing subsequence \( \{a_{n_k}\} \).

The above theorem ensures that the sequence \( \{\sin n\} \) has a monotone subsequence. But finding a specific monotone subsequence is a challenge.

Since every subsequence of a bounded sequence is bounded and every bounded monotone sequence converges, we arrive at the following fundamental result (due to B. Bolzano, 1781–1848, and K. Weierstrass, 1815–1897).
Theorem (Bolzano–Weierstrass)

Let \((a_n)\) be a bounded sequence of real numbers. Then there is a subsequence that converges.

It is instructive to read the alternative proof of this result presented in the textbook, starting with an interval \([-M, M]\) (where \(M > 0\)) that contains all the terms of the sequence and pick any term \(a_{n_1}\), then bisect the interval into two subintervals of equal lengths and from a subinterval of length \(M\) containing infinitely many terms of \((a_n)\), take a term \(a_{n_2}\) with \(n_2 > n_1\). Continue in this fashion to obtain a subsequence \((a_{n_k})\). Note that \(a_{n_k}\) and all subsequent terms in this subsequence are in an interval of length \(4M/2^k\), so the subsequence is Cauchy, and hence, convergent.
A sequence that has a subsequence converging to any number in [0, 1]

Example

Construct a sequence \((a_n)\) such that for each real number \(x\), there is a subsequence of \((a_n)\) that converges to \(x\).

Solution. Since the set \(\mathbb{Q}\) is countable, there is a sequence \((a_n)\) whose terms are all the elements of \(\mathbb{Q}\), with each element of \(\mathbb{Q}\) occurring exactly once. Let \(x\). Choose \(n_1\) such that 
\[|a_{n_1} - x| < 1.\]
Suppose that \(n_1 < n_2 < \cdots < n_t\) have been chosen such that 
\[|a_{n_j} - x| < 1/j\] for all \(j = 1, \ldots, t\). Choose \(n_{t+1}\) such that 
\(n_t < n_{t+1}\) and 
\[|a_{n_{t+1}} - x| < 1/(t + 1),\] which is possible as there are infinitely many rational numbers in \((x - \frac{1}{t+1}, x + \frac{1}{t+1})\).
It is then clear that the subsequence \((a_{n_k})\) converges to \(x\). \(\square\)
Clearly, the sequence \((a_n) = (1 + (-1)^n) = (0, 2, 0, 2, \ldots)\) diverges. A subsequence of this sequence converges if and only if it has only finitely many terms equal to 0 or it has only finitely many terms equal to 2.

**Corollary (Complex Bolzano-Weierstrass Theorem)**

Let \(\{\alpha_j\}\) be a bounded sequence of complex numbers. Then there is a convergent subsequence.

**Proof:** Write \(\alpha_j = a_j + ib_j\), with \(a_j, b_j \in \mathbb{R}\). The fact that \(\{\alpha_j\}\) is bounded implies that \(\{a_j\}\) is bounded. By the Bolzano–Weierstrass theorem, there is a convergent subsequence \(\{a_{j_k}\}\).

Now the sequence \(\{b_{j_k}\}\) is bounded. So it has a convergent subsequence \(\{b_{j_{k_l}}\}\). Then the sequence \(\{\alpha_{j_{k_l}}\}\) is convergent, and is a subsequence of the original sequence \(\{\alpha_j\}\). \(\square\)
**Divergence to $+\infty$ or $-\infty$**

**Definition**

We say that a real sequence *diverges to* $+\infty$ if for every real number $M > 0$, there is a positive integer $N$ such that $a_j > M$ for all $j > N$. In this case, we say that \(\{a_j\}\) has limit $+\infty$ and we write \(\lim_{j \to \infty} a_j = +\infty\), or $a_j \to +\infty$. Divergence to $-\infty$ is defined similarly. Note that

\[
\lim_{j \to \infty} a_j = -\infty \text{ if and only if } \lim_{j \to \infty} -a_j = +\infty.
\]

For example, \(\{n^3 - 20n - 100\} = \{n^3(1 - \frac{20}{n^2} - \frac{100}{n^3})\}\) diverges to $+\infty$, while from calculus, we also know that \(\{\frac{2^n}{n^5}\}\) diverges to $-\infty$. However, the sequence \(\{n + (-1)^n \cdot n\}\) has no infinite limit and no finite limit, although it has a subsequence (using even subscripts only) that diverges to $+\infty$ and it also has subsequence (using odd subscripts only) that converges to 0.
Every monotone sequence has a limit, possibly $\pm \infty$

Proposition

Let $\{a_j\}$ be an increasing sequence of real numbers. Then the sequence has a limit—either a finite number or $+\infty$.

Let $\{b_j\}$ be a decreasing sequence of real numbers. Then the sequence has a limit—either a finite number or $-\infty$.

In the same spirit as the last definition, we also have the following:

Definition

If $S$ is a set of real numbers which is not bounded above, we say that its supremum (or least upper bound) is $+\infty$.

If $T$ is a set of real numbers which is not bounded below, then we say that its infimum (or greatest lower bound) is $-\infty$. 
Negation of $\lim_{n \to \infty} a_n = \alpha$

Let $\alpha \in \mathbb{C}$. It is useful to have precise understanding of the negation of “the sequence $(a_n)$ converges to $\alpha$”. Note that the negation includes two possibilities, either $(a_n)$ diverges, or $(a_n)$ converges to a number different from $\alpha$. But such formulation is not convenient to apply sometimes.

A precise and specific negation of “the sequence $(a_n)$ converges to $\alpha$” comes directly from the $\epsilon$-$N$ definition.

**Fact**

The sequence $(a_n)$ does not converge to a number $\alpha$ if and only if there is a positive number $\epsilon_1 > 0$ such that for each positive integer $N$, there is a positive integer $n > N$ such that $|a_n - \alpha| \geq \epsilon_1$. 

Negation of $\lim_{n \to \infty} a_n = \alpha$; limits of subsequences

In particular, the above fact may be used to show that if a sequence $(a_n)$ does not converge to a number $\alpha$, then $(a_n)$ has a subsequence that does not have any subsequence converging to $\alpha$.

For a given sequence $(a_n)$, how can we construct a sequence $(b_n)$ with the property that each term $a_j$ is the limit of a subsequence of $(b_n)$? Note that the set

$$S = \{a_j + 1/k : j, k \in \mathbb{N}\}$$

is countable, so there is a sequence $(b_n)$ whose terms are all the elements of $S$. It is left as an exercise to verify that $(b_n)$ has the desired property.