Figure: This is your instructor.
We often obtain information about a new sequence by comparison with a sequence that we already know. Thus it is well to have a catalogue of fundamental sequences which provide a basis for comparison.

**Example**

Fix a real number $a$. The sequence $\{a^j\}$ is called a *power sequence*. If $-1 < a < 1$ then the sequence converges to 0. If $a = 1$ then the sequence is a constant sequence and converges to 1. If $a > 1$ then the sequence diverges to $+\infty$. Finally, if $a \leq -1$ then the sequence diverges.
Recall that, in the first lecture, we discussed the existence of $n$th roots of positive real numbers. If $\alpha > 0$, $m \in \mathbb{Z}$, and $n \in \mathbb{N}$ then we may define

$$\alpha^{m/n} = (\alpha^m)^{1/n}.$$ 

Thus we may talk about rational powers of a positive number. Next, if $\beta \in \mathbb{R}$ then we may define

$$\alpha^\beta = \sup\{\alpha^q : q \in \mathbb{Q}, q < \beta\}.$$ 

Thus we can define any real power of a positive real number. The exercises ask you to verify several basic properties of these exponentials.
Lemma

If $\alpha > 1$ is a real number and $\beta > 0$ then $\alpha^\beta > 1$.

Proof: Let $q$ be a positive rational number which is less than $\beta$. Suppose that $q = m/n$, with $m, n$ integers. It is obvious that $\alpha^m > 1$ and hence that $(\alpha^m)^{1/n} > 1$. Since $\alpha^\beta$ majorizes this last quantity, we are done. \qed
Example

Fix a real number $\alpha$ and consider the sequence $\{j^\alpha\}$. If $\alpha > 0$ then it is easy to see that $j^\alpha \to +\infty$: to verify this assertion fix $M > 0$ and take the number $N$ to be the first integer after $M^{1/\alpha}$. If $\alpha = 0$ then $j^\alpha$ is a constant sequence, identically equal to 1. If $\alpha < 0$ then $j^\alpha = 1/j^{-\alpha}$. The denominator of this last expression tends to $+\infty$ hence the sequence $j^\alpha$ tends to 0. □
Example

The sequence \( \{j^{1/j}\} \) converges to 1. In fact, consider the expressions \( \alpha_j = j^{1/j} - 1 > 0 \). We have that

\[ j = (\alpha_j + 1)^j \geq \frac{j(j-1)}{2} (\alpha_j)^2, \]

(the latter being just one term from the binomial expansion). Thus

\[ 0 < \alpha_j \leq \sqrt{2/(j-1)} \]

as long as \( j \geq 2 \). It follows that \( \alpha_j \to 0 \) or \( j^{1/j} \to 1 \). \qed
Example

Let $\alpha$ be a positive real number. Then the sequence $\alpha^{1/j}$ converges to 1. To see this, first note that the case $\alpha = 1$ is trivial, and the case $\alpha > 1$ implies the case $\alpha < 1$ (by taking reciprocals). So we concentrate on $\alpha > 1$. But then we have

$$1 < \alpha^{1/j} < j^{1/j}$$

when $j > \alpha$. Since $j^{1/j}$ tends to 1, an earlier proposition applies and the proof is complete.
Example
Let $\lambda > 1$ and let $\alpha$ be real. Then the sequence

$$\left\{ \frac{j^\alpha}{\lambda^j} \right\}_{j=1}^{\infty}$$

converges to 0.

To see this, fix an integer $k > \alpha$ and consider $j > 2k$. [Notice that $k$ is fixed once and for all but $j$ will be allowed to tend to $+\infty$ at the appropriate moment.] Writing $\lambda = 1 + \mu$, $\mu > 0$, we have that

$$\lambda^j = (1 + \mu)^j > \frac{j(j-1)(j-2)\cdots(j-k+1)}{k(k-1)(k-2)\cdots2\cdot1} \cdot 1^{j-k} \cdot \mu^k.$$
Of course this comes from picking out the $k$th term of the binomial expansion for $(1 + \mu)^j$. Notice that, since $j > 2k$, then each of the expressions $j, (j - 1), \ldots (j - k + 1)$ in the numerator on the right exceeds $j/2$. Thus

$$\lambda j > \frac{j^k}{2^k \cdot k!} \cdot \mu^k$$

and

$$0 < \frac{j^\alpha}{\lambda j} < \frac{j^\alpha}{j^k \cdot \mu^k} = \frac{j^{\alpha-k} \cdot 2^k \cdot k!}{\mu^k}.$$ 

Since $\alpha - k < 0$, the right side tends to 0 as $j \to \infty$. $\blacksquare$
Example

The sequence

\[
\left\{ \left( 1 + \frac{1}{j} \right)^j \right\}
\]

converges. In fact it is increasing and bounded above. Use the Binomial Expansion to prove this assertion. The limit of the sequence is the number that we shall later call \( e \) (in honor of Leonhard Euler, 1707–1783, who first studied it in detail). We shall study this sequence in detail later in the book.
Example

The sequence

\[(1 - \frac{1}{j})^j\]

converges to \(1/e\), where the definition of \(e\) is given in the last example. More generally, the sequence

\[(1 + \frac{x}{j})^j\]

converges to \(e^x\) (here \(e^x\) is defined as in the earlier discussion). \(\square\)