Figure: This is your instructor.
In this section we will use standard summation notation:

\[ \sum_{j=m}^{n} a_j \equiv a_m + a_{m+1} + \cdots + a_n. \]

A series is an infinite sum. One of the most effective ways to handle an infinite process in mathematics is with a limit. This consideration leads to the following definition:
Definition

The formal expression

$$\sum_{j=1}^{\infty} a_j,$$

where the $a_j$s are real or complex numbers, is called a series. For $N = 1, 2, 3, \ldots$, the expression

$$S_N = \sum_{j=1}^{N} a_j = a_1 + a_2 + \ldots + a_N$$

is called the $N$th partial sum of the series.
In case

\[
\lim_{N \to \infty} S_N
\]

exists and is finite we say that the series \textit{converges}. The limit of the partial sums is called the \textit{sum} of the series. If the series does not converge, then we say that the series \textit{diverges}.
Notice that the question of convergence of a series, which should be thought of as an *addition process*, reduces to a question about the *sequence* of partial sums.

**Example**

Consider the series

$$\sum_{j=1}^{\infty} 2^{-j}.$$

The $N$th partial sum for this series is

$$S_N = 2^{-1} + 2^{-2} + \cdots + 2^{-N}.$$
In order to determine whether the sequence \( \{S_N\} \) has a limit, we rewrite \( S_N \) as

\[
S_N = (2^{-0} - 2^{-1}) + (2^{-1} - 2^{-2}) + \ldots + (2^{-N+1} - 2^{-N}).
\]

The expression on the right of the last equation telescopes (i.e., successive pairs of terms cancel) and we find that

\[
S_N = 2^{-0} - 2^{-N}.
\]

Thus

\[
\lim_{N \to \infty} S_N = 2^{-0} = 1.
\]

We conclude that the series converges.
Example

Let us examine the series

\[ \sum_{j=1}^{\infty} \frac{1}{j} \]

for convergence or divergence. (This series is commonly called the *harmonic series* because it describes the harmonics in music.)

Now
\[ S_1 = \frac{1}{2} = \frac{2}{2} \]
\[ S_2 = 1 + \frac{1}{2} = \frac{3}{2} \]
\[ S_4 = 1 + \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) \geq 1 + \frac{1}{2} + \left( \frac{1}{4} + \frac{1}{4} \right) \geq 1 + \frac{1}{2} + \frac{1}{2} = \frac{4}{2} \]
\[ S_8 = 1 + \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) + \left( \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) \geq 1 + \frac{1}{2} + \left( \frac{1}{4} + \frac{1}{4} \right) + \left( \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \right) \]
\[ = \frac{5}{2} \]
In general this argument shows that

\[ S_{2^k} \geq \frac{k + 2}{2}. \]

The sequence of \( S_N \)s is increasing since the series contains only positive terms. The fact that the partial sums \( S_1, S_2, S_4, S_8, \ldots \) increases without bound shows that the entire sequence of partial sums must increase without bound. We conclude that the series diverges. \( \square \)
Just as with sequences, we have a Cauchy criterion for series:

**Proposition**

The series $\sum_{j=1}^{\infty} a_j$ converges if and only if, for every $\epsilon > 0$, there is an integer $N \geq 1$ such that, if $n \geq m > N$, then

$$\left| \sum_{j=m}^{n} a_j \right| < \epsilon.$$  

(*)

The condition (*) is called the Cauchy criterion for series.
Proof: Suppose that the Cauchy criterion holds. Pick $\epsilon > 0$ and choose $N$ so large that (*) holds. If $n \geq m > N$, then

$$|S_n - S_m| = \left| \sum_{j=m+1}^{n} a_j \right| < \epsilon$$

by hypothesis. Thus the sequence \{\(S_N\)\} is Cauchy in the sense discussed for sequences in an earlier lecture. We conclude that the sequence \{\(S_N\)\} converges; by definition, therefore, the series converges.
Conversely, if the series converges then, by definition, the sequence $\{S_N\}$ of partial sums converges. In particular, the sequence $\{S_N\}$ must be Cauchy. Thus, for any $\epsilon > 0$, there is a number $N > 0$ such that if $n \geq m > N$ then

$$|S_n - S_m| < \epsilon.$$ 

This just says that

$$\left| \sum_{j=m+1}^{n} a_j \right| < \epsilon,$$

and this last inequality is the Cauchy criterion for series. \qed
Example

Let us use the Cauchy criterion to verify that the series

$$\sum_{j=1}^{\infty} \frac{1}{j \cdot (j + 1)}$$

converges.

Notice that, if $n \geq m > 1$, then

$$\left| \sum_{j=m}^{n} \frac{1}{j \cdot (j + 1)} \right| = \left( \frac{1}{m} - \frac{1}{m+1} \right) + \left( \frac{1}{m+1} - \frac{1}{m+2} \right) + \cdots + \left( \frac{1}{n} - \frac{1}{n+1} \right).$$
The sum on the right plainly telescopes and we have

\[ \left| \sum_{j=m}^{n} \frac{1}{j \cdot (j + 1)} \right| = \frac{1}{m} - \frac{1}{n + 1}. \]

Let \( \epsilon > 0 \). Let us choose \( N \) to be the next integer after \( 1/\epsilon \). Then, for \( n \geq m > N \), we may conclude that

\[ \left| \sum_{j=m}^{n} \frac{1}{j \cdot (j + 1)} \right| = \frac{1}{m} - \frac{1}{n + 1} < \frac{1}{m} < \frac{1}{N} < \epsilon. \]

This is the desired conclusion. \( \square \)
The next result gives a necessary condition for a series to converge. It is a useful device for detecting divergent series, although it can never tell us that a series converges.

Proposition (The Zero Test)

If the series

\[ \sum_{j=1}^{\infty} a_j \]

converges then the terms \( a_j \) tend to zero as \( j \to \infty \).
Proof: Since we are assuming that the series converges, then it must satisfy the Cauchy criterion. Let $\epsilon > 0$. Then there is an integer $N \geq 1$ such that, if $n \geq m > N$, then

$$\left| \sum_{j=m}^{n} a_j \right| < \epsilon.$$  \hspace{1cm} (***)

We take $n = m$ and $m > N$. Then (***) becomes

$$|a_m| < \epsilon.$$  

But this is precisely the conclusion that we desire. \hspace{1cm} \Box
Example

The series $\sum_{j=1}^{\infty} (-1)^j$ must diverge, *even though its terms appear to be cancelling each other out*. The reason is that the summands do not tend to zero; hence the preceding proposition applies.

Write out several partial sums of this series to see more explicitly that the partial sums are $-1$, $+1$, $-1$, $+1$, $\ldots$ and hence that the series diverges.
We conclude this section with a necessary and sufficient condition for convergence of a series of nonnegative terms. As with some of our other results on series, it amounts to little more than a restatement of a result on sequences.

**Proposition**

A series

\[ \sum_{j=1}^{\infty} a_j \]

with all \( a_j \geq 0 \) is convergent if and only if the sequence of partial sums is bounded.
Proof: Notice that, because the summands are nonnegative, we have

\[ S_1 = a_1 \leq a_1 + a_2 = S_2, \]
\[ S_2 = a_1 + a_2 \leq a_1 + a_2 + a_3 = S_3, \]

and in general

\[ S_N \leq S_N + a_{N+1} = S_{N+1}. \]

Thus the sequence \( \{S_N\} \) of partial sums forms a increasing sequence. We know that such a sequence is convergent to a finite limit if and only if it is bounded above (see our earlier discussion). This completes the proof. \( \square \)
Example

The series $\sum_{j=1}^{\infty} 1$ is divergent since the summands are nonnegative and the sequence of partial sums \( \{S_N\} = \{N\} \) is unbounded.

Referring back to our example on harmonic series, we see that the series $\sum_{j=1}^{\infty} \frac{1}{j}$ diverges because its partial sums are unbounded.

We see from the first example that the series $\sum_{j=1}^{\infty} 2^{-j}$ converges because its partial sums are all bounded above by 1. □
It is frequently convenient to begin a series with summation at $j = 0$ or some other term instead of $j = 1$. All of our convergence results still apply to such a series because of the Cauchy criterion. In other words, the convergence or divergence of a series will depend only on the behavior of its “tail.”