Figure: This is your instructor.
Definition
Let $S$ be a set of real numbers. We say that $S$ is disconnected if it is possible to find a pair of open sets $U$ and $V$ such that

1. $U \cap S \neq \emptyset$, $V \cap S \neq \emptyset$,
2. $(U \cap S) \cap (V \cap S) = \emptyset$, and
3. $S = (U \cap S) \cup (V \cap S)$.

See the figure below. If no such $U$ and $V$ exist, then we say that $S$ is connected.

Figure: The idea of disconnected.
Note that in the above definition, condition (1) means that each of $U$ and $V$ contains some element of $S$, while condition (2) says that no common element of $U$ and $V$ is in $S$. Condition (3) means that $S \subseteq U \cup V$.

Example

The set $T = \{x \in \mathbb{R} : |x| < 1, x \neq 0\}$ is disconnected. Take $U = \{x : x < 0\}$ and $V = \{x : x > 0\}$. Then $U \cap T = \{x : -1 < x < 0\} \neq \emptyset$ and $V \cap T = \{x : 0 < x < 1\} \neq \emptyset$. Also $(U \cap T) \cap (V \cap T) = \emptyset$. Clearly $T = (U \cap T) \cup (V \cap T)$. Hence, $T$ is disconnected.
It is clear that a disconnected set has the property that there are disjoint open sets that, in effect, *disconnect* the set. The next example looks at the contrapositive.

**Example**

The set $X = [-1, 1]$ is connected. To see this, suppose to the contrary that there exist open sets $U$ and $V$ such that $U \cap X \neq \emptyset$, $V \cap X \neq \emptyset$, $(U \cap X) \cap (V \cap X) = \emptyset$, and

$$X = (U \cap X) \cup (V \cap X).$$

Choose $a \in U \cap X$ and $b \in V \cap X$. We may assume $a < b$. Set

$$\alpha = \sup (U \cap [a, b]).$$

Now $[a, b] \subseteq X$ hence $U \cap [a, b]$ is disjoint from $V$. Thus $\alpha \leq b$. But $\bar{V}$ is closed hence $\alpha \notin V$. It follows that $\alpha < b$. 

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If $\alpha \in U$ then, because $U$ is open, there exists an $\tilde{\alpha} \in U$ such that $\alpha < \tilde{\alpha} < b$. This would mean that we chose $\alpha$ incorrectly. Hence $\alpha \notin U$. But $\alpha \notin U$ and $\alpha \notin V$ means $\alpha \notin X$. On the other hand, $\alpha$ is the supremum of a subset of $X$ (since $a \in X$, $b \in X$, and $X$ is an interval). Since $X$ is a closed interval, we conclude that $\alpha \in X$. This contradiction shows that $X$ must be connected. \qed
With small modifications, the discussion in the last example demonstrates that any closed interval is connected (Exercise 1). See figure below. Also (see Exercise 2), we may similarly see that any open interval or half-open interval is connected. In fact the converse is true as well.

**Figure:** A closed interval is connected.
Theorem

A subset $S$ of $\mathbb{R}$ is connected if and only if $S$ is an interval.

Proof: If $S$ is not an interval then there exist $a \in S$, $b \in S$ and a point $t$ between $a$ and $b$ such that $t \notin S$. Define $U = \{x \in \mathbb{R} : x < t\}$ and $V = \{x \in \mathbb{R} : t < x\}$. Then $U$ and $V$ are open and disjoint, $U \cap S \neq \emptyset$, $V \cap S \neq \emptyset$, and

$$S = (U \cap S) \cup (V \cap S).$$

Thus $S$ is disconnected.

If $S$ is an interval then we prove that it is connected using the methodology of Example 4.45.
The Cantor set is not connected; indeed it is disconnected in a special sense. Call a set $S$ **totally disconnected** if, for each distinct $x \in S$, $y \in S$, there exist disjoint open sets $U$ and $V$ such that $x \in U$, $y \in V$, and $S = (U \cap S) \cup (V \cap S)$.

**Proposition**

*The Cantor set is totally disconnected.*
Proof: Let $x, y \in C$ be distinct and assume that $x < y$. Set $\delta = |x - y|$. Choose $j$ so large that $3^{-j} < \delta$. Then $x, y \in S_j$, but $x$ and $y$ cannot both be in the same interval of $S_j$ (since the intervals will have length equal to $3^{-j}$). It follows that there is a point $t$ between $x$ and $y$ that is not an element of $S_j$, hence not an element of $C$. Set $U = \{s : s < t\}$ and $V = \{s : s > t\}$. Then $x \in U \cap C$ hence $U \cap C \neq \emptyset$; likewise, $y \in V \cap C \neq \emptyset$. Also $(U \cap C) \cap (V \cap C) = \emptyset$. Finally $C = (C \cap U) \cup (C \cap V)$. Thus $C$ is totally disconnected. \qed