Figure: This is your instructor.
Today we will have a short lesson on perfect sets. Anyway, this is a good way to end the chapter.

**Definition:** A set $S \subseteq \mathbb{R}$ is called *perfect* if it is closed and if every point of $S$ is an accumulation point of $S$.

The property of being perfect is a rather special one: it means that the set has no isolated points.
**Example:** Consider the set $S = [0, 2]$. This set is perfect. Because (i) it is closed, (ii) any interior point is clearly an accumulation point, (iii) 0 is the limit of $\{1/j\}$ so is an accumulation point, and (iv) 2 is the limit of $\{2 - 1/j\}$ so is an accumulation point.
Clearly any closed interval \([a, b]\) is perfect. After all, a point \(x\) in the interior of the interval is surrounded by an entire open interval \((x - \epsilon, x + \epsilon)\) of elements of the interval; moreover \(a\) is the limit of elements from the right and \(b\) is the limit of elements from the left.
**Example:** The Cantor set, *a totally disconnected set*, is perfect. It is definitely closed. Now fix $x \in C$. Then $x \in S_1$. Thus $x$ is in one of the two intervals composing $S_1$. One (or perhaps both) of the endpoints of that interval does not equal $x$. Call that endpoint $a_1$. Likewise $x \in S_2$. Therefore $x$ lies in one of the intervals of $S_2$. Choose an endpoint $a_2$ of that interval which does not equal $x$. Continuing in this fashion, we construct a sequence $\{a_j\}$. Notice that *each of the elements of this sequence lies in the Cantor set* (why?). Finally, $|x - a_j| \leq 3^{-j}$ for each $j$. Therefore $x$ is the limit of the sequence. We have thus proved that the Cantor set is perfect. $\Box$
The fundamental theorem about perfect sets tells us that such a set must be rather large. We have

**Theorem:** A nonempty perfect set must be uncountable.
**Proof:** Let \( S \) be a nonempty perfect set. Since \( S \) has accumulation points, it cannot be finite. Therefore it is either countable or uncountable.

Seeking a contradiction, we suppose that \( S \) is countable. Write \( S = \{s_1, s_2, \ldots \} \). Set \( U_1 = (s_1 - 1, s_1 + 1) \). Then \( U_1 \) is a neighborhood of \( s_1 \). Now \( s_1 \) is a limit point of \( S \) so there must be infinitely many elements of \( S \) lying in \( U_1 \). We select a bounded open interval \( U_2 \) such that \( \overline{U}_2 \subseteq U_1 \), \( \overline{U}_2 \) does not contain \( s_1 \), and \( U_2 \) does contain some element of \( S \).
Continuing in this fashion, assume that \( s_1, \ldots, s_j \) have been selected and choose a bounded interval \( U_{j+1} \) such that (i) \( \overline{U}_{j+1} \subseteq U_j \), (ii) \( s_j \not\in \overline{U}_{j+1} \), and (iii) \( U_{j+1} \) contains some element of \( S \).

Observe that each set \( V_j = \overline{U}_j \cap S \) is closed and bounded, hence compact. Also each \( V_j \) is nonempty by construction but \( V_j \) does not contain \( s_{j-1} \). It follows that \( V = \cap_j V_j \) cannot contain \( s_1 \) (since \( V_2 \) does not), cannot contain \( s_2 \) (since \( V_3 \) does not), indeed cannot contain any element of \( S \). Hence \( V \), being a subset of \( S \), is empty. But \( V \) is the decreasing intersection of nonempty compact sets, hence cannot be empty!

This contradiction shows that \( S \) cannot be countable. So it must be uncountable. \( \square \)
**Corollary:** If $a < b$ then the closed interval $[a, b]$ is uncountable.
**Proof:** The interval $[a, b]$ is perfect. □
We also have a new way of seeing that the Cantor set is uncountable, since it is perfect:

**Corollary:** *The Cantor set is uncountable.*