Math 4111
November 6, 2020 Lecture

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Figure: This is your instructor.
**Definition:** Let $E \subseteq \mathbb{R}$ be a set and let $f$ be a real-valued function with domain $E$. Fix a point $P$ which is in $E$ and is also an accumulation point of $E$. We say that $f$ is *continuous* at $P$ if

$$\lim_{x \to P} f(x) = f(P).$$
We learned from the penultimate example of the last lecture that polynomial functions are continuous at every real $x$. So are the transcendental functions $\sin x$ and $\cos x$. A rational function is continuous at every point of its domain.
Example: The function

\[ h(x) = \begin{cases} 
\sin(1/x) & \text{if } x \neq 0 \\
1 & \text{if } x = 0 
\end{cases} \]

is discontinuous at 0. See the next figure. The reason is that

\[ \lim_{x \to 0} h(x) \]

does not exist. (Details of this assertion are left for you: notice that \( h(1/(j\pi)) = 0 \) while \( h(2/[(4j + 1)\pi]) = 1 \) for \( j = 1, 2, \ldots \).)
Figure: A function discontinuous at 0.
The function

\[ k(x) = \begin{cases} 
  x \cdot \sin(1/x) & \text{if } x \neq 0 \\
  1 & \text{if } x = 0 
\end{cases} \]

is also discontinuous at \( x = 0 \). This time the limit \( \lim_{x \to 0} k(x) \) exists, but the limit does not agree with \( k(0) \).
However, the function

\[ m(x) = \begin{cases} 
  x \cdot \sin(1/x) & \text{if } x \neq 0 \\
  0 & \text{if } x = 0 
\end{cases} \]

is continuous at \( x = 0 \) because the limit at 0 exists and agrees with the value of the function there. See the next figure. \( \square \)
Figure: A function continuous at 0.
The arithmetic operations $+, -, \times,$ and $\div$ preserve continuity (so long as we avoid division by zero). We now formulate this assertion as a theorem.

**Theorem:** Let $f$ and $g$ be functions with domain $E$ and let $P$ be a point of $E$ which is also an accumulation point of $E$. If $f$ and $g$ are continuous at $P$ then so are $f \pm g$, $f \cdot g$, and (provided $g(P) \neq 0$) $f/g$.

**Proof:** Apply the relevant theorem from the last lecture. \qed
Continuous functions may also be characterized using sequences:

**Proposition:** Let $f$ be a function with domain $E$ and fix $P \in E$ which is an accumulation point of $E$. The function $f$ is continuous at $P$ if and only if, for every sequence $\{a_j\} \subseteq E$ satisfying $\lim_{j \to \infty} a_j = P$, it holds that

$$\lim_{j \to \infty} f(a_j) = f(P).$$
Proof: Apply the appropriate proposition from the last lecture.

Recall that, if \( g \) is a function with domain \( D \) and range \( E \), and if \( f \) is a function with domain \( E \) and range \( F \), then the composition of \( f \) and \( g \) is

\[ f \circ g(x) = f(g(x)) \, . \]

See the figure.
Figure: Composition of functions.
Proposition: Let $g$ have domain $D$ and range $E$ and let $f$ have domain $E$ and range $F$. Let $P \in D$. Suppose that $P$ is an accumulation point of $D$ and $g(P)$ is an accumulation point of $E$. Assume that $g$ is continuous at $P$ and that $f$ is continuous at $g(P)$. Then $f \circ g$ is continuous at $P$. 
**Proof:** Let \( \{a_j\} \) be any sequence in \( D \) such that \( \lim_{j \to \infty} a_j = P \). Then

\[
\lim_{j \to \infty} f \circ g(a_j) = \lim_{j \to \infty} f(g(a_j)) = f \left( \lim_{j \to \infty} g(a_j) \right)
\]

\[
= f \left( g \left( \lim_{j \to \infty} a_j \right) \right) = f(g(P)) = f \circ g(P).
\]

Now apply the last proposition. \( \square \)
**Example:** It is not the case that if

\[ \lim_{x \to P} g(x) = \ell \]

and

\[ \lim_{t \to \ell} f(t) = m \]

then

\[ \lim_{x \to P} f \circ g(x) = m . \]
A counterexample is given by the functions

\[ g(x) = 0 \]

\[ f(x) = \begin{cases} 
2 & \text{if } x \neq 0 \\
5 & \text{if } x = 0.
\end{cases} \]
Notice that \( \lim_{x \to 0} g(x) = 0 \), \( \lim_{t \to 0} f(t) = 2 \), yet \( \lim_{x \to 0} f \circ g(x) = 5 \).

The additional hypothesis that \( f \) be continuous at \( \ell \) is necessary in order to guarantee that the limit of the composition will behave as expected.
Next we explore the topological approach to the concept of continuity. Whereas the analytic approach that we have been discussing so far considers continuity one point at a time, the topological approach considers all points simultaneously. Let us call a function continuous if it is continuous at every point of its domain.

**Definition:** Let $f$ be a function with domain $E$ and let $W$ be any set of real numbers. We define

$$f^{-1}(W) = \{ x \in E : f(x) \in W \}.$$

We sometimes refer to $f^{-1}(W)$ as the *inverse image* of $W$ under $f$. 


Theorem: Let $f$ be a function with domain $E$. The function $f$ is continuous if and only if the inverse image of any open set under $f$ is the intersection of $E$ with an open set.

In particular, if $E$ is open then $f$ is continuous if and only if the inverse image of every open set under $f$ is open.
Proof: Assume that $f$ is continuous. Let $O$ be any open set in $\mathbb{R}$ and let $P \in f^{-1}(O)$. Then, by definition, $f(P) \in O$. Since $O$ is open, there is an $\epsilon > 0$ such that the interval $(f(P) - \epsilon, f(P) + \epsilon)$ lies in $O$. By the continuity of $f$ we may select a $\delta > 0$ such that if $x \in E$ and $|x - P| < \delta$ then $|f(x) - f(P)| < \epsilon$. In other words, if $x \in E$ and $|x - P| < \delta$ then $f(x) \in O$ or $x \in f^{-1}(O)$. Thus we have found an open interval $I = (P - \delta, P + \delta)$ about $P$ whose intersection with $E$ is contained in $f^{-1}(O)$. So $f^{-1}(O)$ is the intersection of $E$ with an open set.
Conversely, suppose that for any open set $O \subseteq \mathbb{R}$ we have that $f^{-1}(O)$ is the intersection of $E$ with an open set. Fix $P \in E$. Choose $\epsilon > 0$. Then the interval $(f(P) - \epsilon, f(P) + \epsilon)$ is an open set. By hypothesis the set $f^{-1}((f(P) - \epsilon, f(P) + \epsilon))$ is the intersection of $E$ with an open set.
This set contains the point $P$. Thus there is a $\delta > 0$ such that

$$E \cap (P - \delta, P + \delta) \subseteq f^{-1}((f(P) - \epsilon, f(P) + \epsilon)).$$

But that just says that

$$f(E \cap (P - \delta, P + \delta)) \subseteq (f(P) - \epsilon, f(P) + \epsilon).$$

In other words, if $|x - P| < \delta$ and $x \in E$ then $|f(x) - f(P)| < \epsilon$. But that means that $f$ is continuous at $P$. \qed
Example: Since any open subset of the real numbers is a countable or finite disjoint union of intervals then—in order to check that the inverse image under a function $f$ of every open set is open—it is enough to check that the inverse image of any open interval is open. This is frequently easy to do.

For example, if $f(x) = x^2$ then the inverse image of an open interval $(a, b)$ is $(-\sqrt{b}, -\sqrt{a}) \cup (\sqrt{a}, \sqrt{b})$ if $a > 0$, is $(-\sqrt{b}, \sqrt{b})$ if $a \leq 0$, $b \geq 0$, and is $\emptyset$ if $a < b < 0$. Thus the function $f$ is continuous.

Note that, by contrast, it is somewhat tedious to give an $\varepsilon - \delta$ proof of the continuity of $f(x) = x^2$. 

□
Corollary: Let $f$ be a function with domain $E$. The function $f$ is continuous if and only if the inverse image of any closed set $F$ under $f$ is the intersection of $E$ with some closed set.

In particular, if $E$ is closed then $f$ is continuous if and only if the inverse image of any closed set $F$ under $f$ is closed.
Proof: It is enough to prove that

\[ f^{-1}(cF) = c(f^{-1}(F)) \, . \]

We leave this assertion as an exercise for you. \( \square \)