Figure: This is your instructor.
l’Hôpital’s Rule (actually due to his teacher J. Bernoulli (1667–1748)) is a useful device for calculating limits, and a nice application of the Cauchy Mean Value Theorem. Here we present a special case of the theorem.
Theorem: Suppose that \( f \) and \( g \) are differentiable functions on an open interval \( I \) and that \( p \in I \). If \( \lim_{x \to p} f(x) = \lim_{x \to p} g(x) = 0 \) and if
\[
\lim_{x \to p} \frac{f'(x)}{g'(x)}
\]
equals a real number \( \ell \) then
\[
\lim_{x \to p} \frac{f(x)}{g(x)} = \ell.
\]
**Proof:** Fix a real number $a > \ell$. By the formula there is a number $q > p$ such that, if $p < x < q$, then

$$\frac{f'(x)}{g'(x)} < a.$$ 

But now, if $p < s < t < q$, then

$$\frac{f(t) - f(s)}{g(t) - g(s)} = \frac{f'(x)}{g'(x)}$$

for some $s < x < t$ (by Cauchy’s Mean Value Theorem). It follows then from the equation above that

$$\frac{f(t) - f(s)}{g(t) - g(s)} < a.$$
Now let \( s \to p \) and invoke the hypothesis about the zero limit of \( f \) and \( g \) at \( p \) to conclude that

\[
\frac{f(t)}{g(t)} \leq a
\]

when \( p < t < q \). Since \( a \) is an arbitrary number to the right of \( \ell \) we conclude that

\[
\limsup_{t \to p^+} \frac{f(t)}{g(t)} \leq \ell.
\]

Similar arguments show that

\[
\liminf_{t \to p^+} \frac{f(t)}{g(t)} \geq \ell; \\
\limsup_{t \to p^-} \frac{f(t)}{g(t)} \leq \ell; \\
\liminf_{t \to p^-} \frac{f(t)}{g(t)} \geq \ell.
\]

We conclude that the desired limit exists and equals \( \ell \). \( \square \)
A closely related result, with a similar proof, is this:

**Theorem:** Suppose that $f$ and $g$ are differentiable functions on an open interval $I$ and that $p \in I$. If

$$\lim_{x \to p} f(x) = \lim_{x \to p} g(x) = \pm \infty \quad \text{and if}$$

$$\lim_{x \to p} \frac{f'(x)}{g'(x)}$$

exists and equals a real number $\ell$ then

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \ell.$$
**Example:** Let

\[ f(x) = \ln|x|^{(x^2)}. \]

We wish to determine \( \lim_{x \to 0} f(x) \). To do so, we define

\[ F(x) = \ln f(x) = x^2 \ln|\ln|x|| = \frac{\ln|\ln|x||}{1/x^2}. \]

Notice that both the numerator and the denominator tend to \( \pm \infty \) as \( x \to 0 \). So the hypotheses of l’Hôpital’s rule are satisfied and the limit is

\[
\lim_{x \to 0} \frac{\ln|\ln|x||}{1/x^2} = \lim_{x \to 0} \frac{(\ln|\ln|x|)'}{(1/x^2)'} = \lim_{x \to 0} \frac{1/[x \ln|x|]}{-2/x^3} = \lim_{x \to 0} \frac{-x^2}{2 \ln|x|} = 0.
\]

Since \( \lim_{x \to 0} F(x) = 0 \) we may calculate that the original limit has value \( \lim_{x \to 0} f(x) = 1. \)
Proposition: Let $f$ be an invertible function on an interval $(a, b)$ with nonzero derivative at a point $x \in (a, b)$. Let $X = f(x)$. Then $(f^{-1})'(X)$ exists and equals $1/f'(x)$. 
Proof: Observe that, for $T \neq X$,

$$
\frac{f^{-1}(T) - f^{-1}(X)}{T - X} = \frac{1}{\frac{f(t)-f(x)}{t-x}},
$$

where $T = f(t)$. Since $f'(x) \neq 0$, the difference quotients for $f$ in the denominator are bounded from zero hence the limit of the formula exists. This proves that $f^{-1}$ is differentiable at $X$ and that the derivative at that point equals $1/f'(x)$.  

\[\square\]
**Example:** We know that the function $f(x) = x^k$, $k$ a positive integer, is one-to-one and differentiable on the interval $(0, 1)$. Moreover the derivative $k \cdot x^{k-1}$ never vanishes on that interval. Therefore the proposition applies and we find for $X \in (0, 1) = f((0, 1))$ that

\[
(f^{-1})'(X) = \frac{1}{f'(x)} = \frac{1}{f'(X^{1/k})}
\]

\[
= \frac{1}{k \cdot X^{1 - 1/k}} = \frac{1}{k} \cdot X^{1/k - 1}.
\]

In other words,

\[
\left( X^{1/k} \right)' = \frac{1}{k} X^{1/k - 1}.
\]

\[\square\]
We conclude this lecture by saying a few words about higher derivatives. If $f$ is a differentiable function on an open interval $I$ then we may ask whether the function $f'$ is differentiable. If it is, then we denote its derivative by

\[ f'' \text{ or } f^{(2)} \text{ or } \frac{d^2}{dx^2} f \text{ or } \frac{d^2 f}{dx^2}, \]

and call it the second derivative of $f$. Likewise the derivative of the $(k - 1)$th derivative, if it exists, is called the $k$th derivative and is denoted

\[ f''...' \text{ or } f^{(k)} \text{ or } \frac{d^k}{dx^k} f \text{ or } \frac{d^k f}{dx^k}. \]

Observe that we cannot even consider whether $f^{(k)}$ exists at a point unless $f^{(k-1)}$ exists in a neighborhood of that point.
If $f$ is $k$ times differentiable on an open interval $I$ and if each of the derivatives $f^{(1)}, f^{(2)}, \ldots, f^{(k)}$ is continuous on $I$ then we say that the function $f$ is $k$ times continuously differentiable on $I$. We write $f \in C^k(I)$. Obviously there is some redundancy in this definition since the continuity of $f^{(j-1)}$ follows from the existence of $f^{(j)}$. Thus only the continuity of the last derivative $f^{(k)}$ need be checked. Continuously differentiable functions are useful tools in analysis. We denote the class of $k$ times continuously differentiable functions on $I$ by $C^k(I)$. 

Example: For $k = 1, 2, \ldots$ the function

$$f_k(x) = \begin{cases} x^{k+1} & \text{if } x \geq 0 \\ -x^{k+1} & \text{if } x < 0 \end{cases}$$

will be $k$ times continuously differentiable on $\mathbb{R}$ but will fail to be $k + 1$ times differentiable at $x = 0$. More dramatically, an analysis similar to the one we used on the Weierstrass nowhere differentiable function shows that the function

$$g_k(x) = \sum_{j=1}^{\infty} \frac{3^j}{4^{j+jk}} \sin(4^j x)$$

is $k$ times continuously differentiable on $\mathbb{R}$ but will not be $k + 1$ times differentiable at any point (this function, with $k = 0$, was Weierstrass’s original example).
A more refined notion of smoothness/continuity of functions is that of Hölder continuity or Lipschitz continuity (see an earlier lecture). If $f$ is a function on an open interval $I$ and if $0 < \alpha \leq 1$ then we say that $f$ satisfies a *Lipschitz condition* of order $\alpha$ on $I$ if there is a constant $M$ such that for all $s, t \in I$ we have

$$|f(s) - f(t)| \leq M \cdot |s - t|^{\alpha}.$$ 

Such a function is said to be of class $\text{Lip}_\alpha(I)$. Clearly a function of class $\text{Lip}_\alpha$ is uniformly continuous on $I$. For, if $\epsilon > 0$, then we may take $\delta = (\epsilon/M)^{1/\alpha}$: it follows that, for $|s - t| < \delta$, we have

$$|f(s) - f(t)| \leq M \cdot |s - t|^{\alpha} < M \cdot \epsilon/M = \epsilon.$$
Interestingly, when \( \alpha > 1 \) the class \( \text{Lip}_\alpha \) contains only constant functions. For in this instance the inequality

\[
|f(s) - f(t)| \leq M \cdot |s - t|^{\alpha}
\]

leads to

\[
\left| \frac{f(s) - f(t)}{s - t} \right| \leq M \cdot |s - t|^{\alpha - 1}.
\]

Because \( \alpha - 1 > 0 \), letting \( s \to t \) yields that \( f'(t) \) exists for every \( t \in I \) and equals 0. It follows from a corollary in the last lecture that \( f \) is constant on \( I \).
Instead of trying to extend the definition of Lip$_\alpha$(I) to $\alpha > 1$ it is customary to define classes of functions $C^{k,\alpha}$, for $k = 0, 1, \ldots$ and $0 < \alpha \leq 1$, by the condition that $f$ be of class $C^k$ on $I$ and that $f^{(k)}$ be an element of Lip$_\alpha$(I). We leave it as an exercise for you to verify that $C^{k,\alpha} \subseteq C^{\ell,\beta}$ if either $k > \ell$ or both $k = \ell$ and $\alpha \geq \beta$.

In more advanced studies in analysis, it is appropriate to replace Lip$_1$(I), and more generally $C^{k,1}$, with another space (invented by Antoni Zygmund, 1900–1992) defined in a more subtle fashion using second differences as in an example at the end of the last lecture. These matters exceed the scope of this book, but we shall make a few remarks about them in the exercises.