Figure: This is your instructor.
In general, limits do not commute. Since the integral is defined with a limit, and since we saw in the last section that integrals do not always respect limits of functions, we know some concrete instances of non-commutation of limits. The fact that continuity is defined with a limit, and that the limit of continuous functions need not be continuous, gives even more examples of situations in which limits do not commute. Let us now turn to a situation in which limits do commute:
Theorem: Fix a set $S$ and a point $s \in S$. Assume that the functions $f_j$ converge uniformly on the domain $S \setminus \{s\}$ to a limit function $f$. Suppose that each function $f_j(x)$ has a limit as $x \to s$. Then $f$ itself has a limit as $x \to s$ and

$$\lim_{x \to s} f(x) = \lim_{j \to \infty} \lim_{x \to s} f_j(x).$$

Because of the way that $f$ is defined, we may rewrite this conclusion as

$$\lim_{x \to s} \lim_{j \to \infty} f_j(x) = \lim_{j \to \infty} \lim_{x \to s} f_j(x).$$

In other words, the limits $\lim_{x \to s}$ and $\lim_{j \to \infty}$ commute.
Proof: Let \( \alpha_j = \lim_{x \to s} f_j(x) \). Let \( \epsilon > 0 \). There is a number \( N > 0 \) (independent of \( x \in S \setminus \{s\} \)) such that \( j > N \) implies that
\[ |f_j(x) - f(x)| < \frac{\epsilon}{4}. \]
Fix \( j, k > N \). Choose \( \delta > 0 \) such that
\[ 0 < |x - s| < \delta \]
implies both that
\[ |f_j(x) - \alpha_j| < \frac{\epsilon}{4} \]
and
\[ |f_k(x) - \alpha_k| < \frac{\epsilon}{4}. \]
Then
\[ |\alpha_j - \alpha_k| \leq |\alpha_j - f_j(x)| + |f_j(x) - f(x)| + |f(x) - f_k(x)| + |f_k(x) - \alpha_k|. \]
The first and last expressions are less than \( \frac{\epsilon}{4} \) by the choice of \( x \). The middle two expressions are less than \( \frac{\epsilon}{4} \) by the choice of \( N \) (and therefore of \( j \) and \( k \)). We conclude that the sequence \( \alpha_j \) is Cauchy. Let \( \alpha \) be the limit of that sequence.
Letting \( k \to \infty \) in the inequality

\[ |\alpha_j - \alpha_k| < \epsilon \]

that we obtained above yields

\[ |\alpha_j - \alpha| \leq \epsilon \]

for \( j > N \). Now, with \( \delta \) as above and \( 0 < |x - s| < \delta \), we have

\[ |f(x) - \alpha| \leq |f(x) - f_j(x)| + |f_j(x) - \alpha_j| + |\alpha_j - \alpha| . \]

By the choices we have made, the first term is less than \( \epsilon/4 \), the second is less than \( \epsilon/4 \), and the third is less than or equal to \( \epsilon \). Altogether, if \( 0 < |x - s| < \delta \) then \( |f(x) - \alpha| < 2\epsilon \). This is the desired conclusion. \( \square \)
**Example:** Consider the example

\[ f_j(x) = x^j \]

on the interval \([0, 1]\). We see that

\[ \lim_{j \to \infty} f_j(x) = 0 \equiv f(x) \]

for \(0 \leq x < 1\). Thus

\[ \lim_{x \to 1^-} f(x) = 0. \]

But

\[ \lim_{j \to \infty} \lim_{x \to 1^-} f_j(x) = \lim_{j \to \infty} 1 = 1. \]
Thus the two dual limits in the theorem are unequal in this example. But of course the functions $f_j$ do not converge uniformly.
Parallel with our notion of Cauchy sequence of numbers, we have a concept of Cauchy sequence of functions in the uniform sense:

**Definition:** A sequence of functions $f_j$ on a domain $S$ is called a uniformly Cauchy sequence if, for each $\epsilon > 0$, there is an $N > 0$ such that, if $j, k > N$, then

$$|f_j(x) - f_k(x)| < \epsilon \quad \forall x \in S.$$
The key point for “uniformly Cauchy” sequence of functions is that the choice of $N$ does not depend on $x$.

**Proposition:** A sequence of functions $f_j$ is uniformly Cauchy on a domain $S$ if and only if the sequence converges uniformly to a limit function $f$ on the domain $S$. 
Proof: The proof is straightforward and is assigned as an exercise.

We will use the last two results in our study of the limits of differentiable functions. First we consider an example.
Example: Define the function

\[ f_j(x) = \begin{cases} 
0 & \text{if } x \leq 0 \\
jx^2 & \text{if } 0 < x \leq 1/(2j) \\
x - 1/(4j) & \text{if } 1/(2j) < x < \infty 
\end{cases} \]

We leave it as an exercise for you to check that the functions \( f_j \) converge uniformly on the entire real line to the function

\[ f(x) = \begin{cases} 
0 & \text{if } x \leq 0 \\
x & \text{if } x > 0 
\end{cases} \]

(draw a sketch to help you see this).
Notice that each of the functions $f_j$ is continuously differentiable on the entire real line, but $f$ is not differentiable at 0.

It turns out that we must strengthen our convergence hypotheses if we want the limit process to respect differentiation. The basic result is this:
Theorem: Suppose that a sequence $f_j$ of differentiable functions on an open interval $I$ converges pointwise to a limit function $f$. Suppose further that the sequence $f'_j$ converges uniformly on $I$ to a limit function $g$. Then the limit function $f$ is differentiable on $I$ and $f'(x) = g(x)$ for all $x \in I$. 
**Proof:** Let $\epsilon > 0$. The sequence $\{f_j^\prime\}$ is uniformly Cauchy. Therefore we may choose $N$ so large that $j, k > N$ implies that

$$|f'_j(x) - f'_k(x)| < \frac{\epsilon}{2} \quad \forall x \in I.$$ 

Fix a point $P \in I$. Define

$$\mu_j(x) = \frac{f_j(x) - f_j(P)}{x - P}$$

for $x \in I, x \neq P$. It is our intention to apply the theorem above to the functions $\mu_j$. 
First notice that, for each $j$, we have

$$\lim_{x \to P} \mu_j(x) = f'_j(P).$$

Thus

$$\lim_{j \to \infty} \lim_{x \to P} \mu_j(x) = \lim_{j \to \infty} f'_j(P) = g(P).$$

That calculates the limits in one order.
On the other hand,

$$\lim_{j \to \infty} \mu_j(x) = \frac{f(x) - f(P)}{x - P} \equiv \mu(x)$$

for $x \in I \setminus \{P\}$. If we can show that this convergence is uniform then the theorem applies and we may conclude that

$$\lim_{x \to P} \mu(x) = \lim_{j \to \infty} \lim_{x \to P} \mu_j(x) = \lim_{j \to \infty} f'_j(P) = g(P).$$

But this just says that $f$ is differentiable at $P$ and the derivative equals $g$. That is the desired result.
To verify the uniform convergence of the $\mu_j$, we apply the Mean Value Theorem to the function $f_j - f_k$. For $x \neq P$ we have

$$|\mu_j(x) - \mu_k(x)| = \frac{1}{|x - P|} \cdot |(f_j(x) - f_k(x)) - (f_j(P) - f_k(P))|$$

$$= \frac{1}{|x - P|} \cdot |x - P| \cdot |(f_j - f_k)'(\xi)|$$

$$= |(f_j - f_k)'(\xi)|$$

for some $\xi$ between $x$ and $P$. But the displayed line at the beginning of the proof guarantees that the last line does not exceed $\epsilon/2$. That shows that the $\mu_j$ converge uniformly and concludes the proof. \qed
Remark: A little additional effort shows that we need only assume in the theorem that the functions \( f_j \) converge at a single point \( x_0 \) in the domain. One of the exercises asks you to prove this assertion.

Notice further that, if we make the additional assumption that each of the functions \( f'_j \) is continuous, then the proof of the theorem becomes much easier. For then

\[
f_j(x) = f_j(x_0) + \int_{x_0}^{x} f'_j(t) \, dt
\]

by the Fundamental Theorem of Calculus. The hypothesis that the \( f'_j \) converge uniformly then implies that the integrals converge to

\[
\int_{x_0}^{x} g(t) \, dt.
\]
The hypothesis that the functions $f_j$ converge at $x_0$ then allows us to conclude that the sequence $f_j(x)$ converges for every $x$ to $f(x)$ and

$$f(x) = f(x_0) + \int_{x_0}^{x} g(t) \, dt.$$ 

The Fundamental Theorem of Calculus then yields that $f' = g$ as desired.
**Example:** Consider the sequence of functions $f_j(x) = j^{-1/2} \sin(jx)$. This sequence converges uniformly to the identically zero function $f(x) \equiv 0$. But $f'_j = j^{1/2} \cos(jx)$ does not converge at any point.

We can sum up this result by saying that

$$
\lim_{j \to \infty} \frac{d}{dx} f_j(x) \neq \frac{d}{dx} \lim_{j \to \infty} f_j(x).
$$

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