FINAL EXAM

**General Instructions:** Read the statement of each problem carefully. If you want full credit on a problem then you must show your work. If you only write the answer then you will not receive full credit.

You will submit your work on the CrowdMark system. Be sure to answer each question on a separate sheet of paper. Scan in each piece of paper separately.

This exam is worth 100 points. It is 25% of your grade.

Be sure to ask questions if anything is unclear.

(6 points) **1.** Prove that the field of complex numbers cannot be made into an ordered field.

(6 points) **2.** Let $S$ be a set of real numbers which is bounded above. Let $\alpha$ be the supremum of $S$. TRUE OR FALSE: For each $\epsilon > 0$, there is an $s \in S$ such that $\alpha - s < \epsilon$.

(9 points) **3.** Prove that the series 
\[ \sum_{j=1}^{\infty} \frac{\sin^2 j}{j} \]

diverges.

(9 points) **4.** Discuss convergence or divergence of the sequence 
\[ \left\{ \left( \frac{1}{j} \right)^{1/j} \right\}_{j=1}^{\infty} \].
5. Discuss convergence or divergence of the series
\[ \sum_{j=3}^{\infty} \frac{1}{j \log^3 j}. \]

6. Let each \( a_j \) be positive and assume that \( \sum a_j \) converges. Prove that \( \sum j \cdot a_j \) may or may not converge.

7. Let \( f \) be a function that is continuously differentiable on the interval \((0, \infty)\). Assume that
\[ 0 < f'(x) \leq f(x) \]
for all \( x \in (0, \infty) \). Prove that \( f(x) \leq e^x \).

8. Let \( f \) be a twice continuously differentiable function on the real line. Assume that \( f''(x) \geq c > 0 \) for all \( x \) and for some positive constant \( c \). Prove that \( f \) is unbounded above.

9. The union of infinitely many closed sets need not be closed. It need not be open either. Give examples to illustrate the possibilities.

10. Let \( \{p_j\}_{j=1}^{\infty} \) be a sequence of polynomials. Assume that \( p_j(x) \to f(x) \) for every \( x \) in the interval \([0, 1]\) for some function \( f \). Prove that \( f \) need not be a polynomial.

11. Let \( f : [0, 1] \to \mathbb{R} \) be a continuous function and let \( f_n : [0, 1] \to \mathbb{R} \) be a sequence of continuous functions. If \( \{f_n\} \) converges pointwise to \( f \) and if \( f_n(x) \geq f_{n+1}(x) \) for all \( x \) and all \( n \), then prove that \( \{f_n\} \) converges uniformly to \( f \).

12. Let \( \alpha(x) \) be the greatest integer function: The value of \( \alpha(x) \) is the greatest integer less than or equal to \( x \). Calculate
\[ \int_0^5 x^2 \, d\alpha(x). \]

13. Let \( f(x) = \sin x \) and \( g(x) = x^2 \). Calculate
\[ \int_0^1 f(x) \, dg(x). \]
(6 points) 14. Prove that the series
\[ \sum_{j=1}^{\infty} \frac{\sin jx}{j^2} \]
converges uniformly to a continuous function on the interval \([0, 1]\).