Math 4111 Fall, 2020
Krantz

FINAL EXAM

General Instructions: Read the statement of each problem carefully. If you want full credit on a problem then you must show your work. If you only write the answer then you will *not* receive full credit.

You will submit your work on the CrowdMark system. Be sure to answer each question on a separate sheet of paper. Scan in each piece of paper separately.

This exam is worth 100 points. It is 25% of your grade.

Be sure to ask questions if anything is unclear.

(6 points) **1.** Prove that the field of complex numbers *cannot* be made into an ordered field.

(6 points) **2.** Let S be a set of real numbers which is bounded above. Let α be the supremum of S. TRUE OR FALSE: For each $\epsilon > 0$, there is an $s \in S$ such that $\alpha - s < \epsilon$.

(9 points) **3.** Prove that the series

$$\sum_{j=1}^{\infty} \frac{\sin^2 j}{j}$$

diverges.

(9 points) **4.** Discuss convergence or divergence of the sequence

$$\left\{ \left(\frac{1}{j} \right)^{1/j} \right\}_{j=1}^{\infty} .$$

(6 points) **5.** Discuss convergence or divergence of the series

$$\sum_{j=3}^{\infty} \frac{1}{j \log^3 j}\,.$$

(6 points) **6.** Let each a_j be positive and assume that $\sum_j a_j$ converges. Prove that $\sum_j j \cdot a_j$ may or may not converge.

(8 points) **7.** Let f be a function that is continuously differentiable on the interval $(0, \infty)$. Assume that

$$0 < f'(x) \leq f(x)$$

for all $x \in (0, \infty)$. Prove that $f(x) \leq e^x$.

(6 points) **8.** Let f be a twice continuously differentiable function on the real line. Assume that $f''(x) \geq c > 0$ for all x and for some positive constant c. Prove that f is unbounded above.

(8 points) **9.** The union of infinitely many closed sets need not be closed. It need not be open either. Give examples to illustrate the possibilities.

(8 points) **10.** Let $\{p_j\}_{j=1}^{\infty}$ be a sequence of polynomials. Assume that $p_j(x) \to f(x)$ for every x in the interval $[0, 1]$ for some function f. Prove that f need not be a polynomial.

(8 points) **11.** Let $f : [0, 1] \to \mathbb{R}$ be a continuous function and let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of continuous functions. If $\{f_n\}$ converges pointwise to f and if $f_n(x) \geq f_{n+1}(x)$ for all x and all n, then prove that $\{f_n\}$ converges uniformly to f.

(7 points) **12.** Let $\alpha(x)$ be the greatest integer function: The value of $\alpha(x)$ is the greatest integer less than or equal to x. Calculate

$$\int_0^5 x^2 \, d\alpha(x)\,.$$

(7 points) **13.** Let $f(x) = \sin x$ and $g(x) = x^2$. Calculate

$$\int_0^1 f(x) \, dg(x)\,.$$

(6 points) **14.** Prove that the series

$$\sum_{j=1}^{\infty} \frac{\sin jx}{j^2}$$

converges uniformly to a continuous function on the interval $[0, 1]$.