PRACTICE EXAM FOR FINAL EXAM

(6 points) 1. Prove that multiplication is commutative in the complex numbers.

(6 points) 2. Let $S$ be a set of real numbers. Prove that $\text{sup } S$ and $\text{inf } S$ are unique.

(9 points) 3. Prove that the series

$$
\sum_{j=1}^{\infty} \frac{\cos^2 j}{j}
$$

diverges.

(9 points) 4. Discuss convergence or divergence of the sequence

$$
\left\{ \frac{j!}{2^j} \right\}_{j=1}^{\infty}
$$

(6 points) 5. Discuss convergence or divergence of the series

$$
\sum_{j=3}^{\infty} \frac{1}{j \log j}
$$

(6 points) 6. Let each $a_j$ be positive and assume that $\sum_j a_j$ diverges. Prove that $\sum_j a_j/j^{1/2}$ may or may not converge.

(8 points) 7. Let $f$ be a function that is twice continuously differentiable on the interval $(0, \infty)$. Assume that

$$
0 < f''(x) \leq x
$$

for all $x \in (0, \infty)$. What can you say about the growth of $f$ at infinity?
8. Let $f$ be a twice continuously differentiable function on the real line. Assume that $f''(x) \leq -c < 0$ for all $x$ and for some positive constant $c$. Prove that $f$ is unbounded below.

9. The intersection of infinitely many open sets need not be open. It need not be closed either. Give examples to illustrate the possibilities.

10. Let $\{p_j\}_{j=1}^{\infty}$ be a collection of polynomials. Assume that $\sum_j p_j(x) = f(x)$ for every $x$ in the interval $[0, 1]$ for some function $f$. Prove that $f$ need not be a polynomial.

11. Prove that the uniform limit of continuously differentiable functions need not be continuously differentiable.

12. Let $\alpha(x)$ be the greatest integer function: The value of $\alpha(x)$ is the greatest integer less than or equal to $x$. Calculate

$$\int_0^5 \cos x \, d\alpha(x).$$

13. Let $f(x) = x^2$ and $g(x) = \sin x$. Calculate

$$\int_0^1 f(x) \, dg(x).$$

14. Prove that the series

$$\sum_{j=1}^{\infty} \frac{\cos jx}{j^3}$$

converges uniformly to a continuous function on the interval $[0, 1]$. 
