MATH 4211

Solutions to Practice Final Exam

1. Let $z = x + iy$, $w = u + iv$. Then

$$z, w = (xu - yv) + i(xv + yu)$$

$$w, z = (ux - vy) + i(uy + vx).$$

These are equal.

2. Let $S$ be bounded above and let $\sigma$ be a supremum of $S$. If $\tau$ is another supremum and $\tau \neq \sigma$, say that $\tau < \sigma$.

This would mean that $\sigma$ is not the least upper bound. Contradiction. Same contradiction if $\sigma < \tau$. So $\sigma = \tau$.

Same reasoning for the minimum.

3. \[
\cos^2 \frac{\pi}{j} = \frac{1 + \cos \frac{2\pi}{j}}{2}. \text{ The series } \sum \frac{\cos^2 \frac{\pi}{j}}{j} \text{ converges by Abel's Test as in Example 3.35.}
\]

And the series $\sum \frac{1}{j}$ diverges. So $\sum \frac{\cos^2 \frac{\pi}{j}}{j}$ diverges.

4. For $j \geq 9$,

$$\frac{j!}{2^j} \geq \frac{j(j-1) \cdots 9}{2^j} > \left(\frac{2}{3}\right)^{j-8} - 2^{j-24}$$

as $j \to \infty$. 
5. By the integral test,
\[
\int_{1}^{\infty} \frac{1}{x \log x} \, dx = \log \log x \bigg|_{1}^{\infty} = \infty
\]
so the integral diverges and series diverges.

6. The series \( \sum_{j=2}^{\infty} \frac{1}{j^{3/4}} \) diverges but \( \sum_{j=2}^{\infty} \frac{1}{j^{1/2}} = \sum_{j=2}^{\infty} \frac{1}{j^{1/3}} \) converges.

The series \( \sum_{j=2}^{\infty} \frac{1}{j^{1/2}} \) diverges and \( \sum_{j=1}^{\infty} \frac{1}{j} \) also diverges.

7. \( f''(x) \leq x \)

\[
\int_{0}^{t} f''(x) \, dx \leq \int_{0}^{t} x \, dx
\]

\[
f''(t) - f(0) \leq \frac{t^2}{2}
\]

\[
\int_{0}^{w} f''(t) - f(0) \, dt \leq \int_{0}^{w} \frac{t^2}{2} \, dt
\]

\[
f(w) - f(0)w - f'(0) \leq \frac{w^3}{6}
\]

\[
f(w) \leq f(0)w + f'(0) + \frac{w^3}{6}
\]

So \( f \) grows at most cubically at \( \infty \).
8. Let \( f(x) = \sqrt{x} \). So we are looking at
\[
f(x+1) - f(x) = (x+1) - x, \quad f'(x) \quad (\star)
\]
for some \( \xi \) between \( x \) and \( x+1 \).

But \( f'(x) = \frac{1}{2} x^{-\frac{1}{2}} \), so
\[
|f'(\xi)| \leq \frac{1}{2} (x^{-\frac{1}{2}}).
\]

Hence
\[
|f(x+1) - f(x)| \leq 1 \cdot \frac{1}{2} (x^{\frac{1}{2}}) \to 0
\]
as \( x \to +\infty \).

9. Let \( O_j = (\frac{-1}{2^{j+1}}, \frac{1}{2^j}) \), \( j = 2, 3, \ldots \).

Then \( O_j = [0, 1] \) which is closed.

Let \( U_j = (\frac{1}{2^j}, 1) \), \( j = 2, 3, \ldots \).

Then \( \cap O_j = [0, 1] \) which is half-open.

Let \( V_j = (\frac{1}{j}, \infty) \), \( j = 2, 3, \ldots \).

Then \( \cap V_j = (1, \infty) \) which is open.

20. Let \( p_j = \frac{(-1)^{j+1} x^{2j+1}}{(2j+1)!} \). These are polynomials.

But \( \sum_{j=0}^{\infty} p_j = \sin x \), which is not a polynomial.
11. Let
\[ f_j(x) = \begin{cases} \frac{1}{\sqrt{j}} + \frac{1}{j} + 2(x - \frac{1}{\sqrt{j}}) & \text{if } x \geq \frac{1}{\sqrt{j}} \\ \frac{1}{\sqrt{j}}x^2 + \frac{1}{j} & \text{if } -\frac{1}{\sqrt{j}} < x < \frac{1}{\sqrt{j}} \\ \frac{1}{\sqrt{j}} + \frac{1}{j} - 2(x + \frac{1}{\sqrt{j}}) & \text{if } x \leq -\frac{1}{\sqrt{j}} \end{cases} \]

Then \( f_j \) is obviously continuous and
\[ f_j \to f = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases} \]
uniformly on \([-1, 1]\).

Also
\[ f_j'(x) = \begin{cases} 2 & \text{if } x \geq \frac{1}{\sqrt{j}} \\ \frac{2}{\sqrt{j}}x & \text{if } -\frac{1}{\sqrt{j}} < x < \frac{1}{\sqrt{j}} \\ -2 & \text{if } x \leq -\frac{1}{\sqrt{j}} \end{cases} \]
is continuous on \([-1, 1]\). But \( f \) is not differentiable at 0.

12. We may as well use a partition with mesh less than \( L \). The only intervals that are non-zero in the calculation of the
Riemann-Stieltjes sum. Note that contain 1, 2, 3, 4, 5. And then \( \Delta x_j = 1 \).
So the value of the integral is
\[
\cos 1 + \cos 2 + \cos 3 + \cos 4 + \cos 5.
\]

13. \[\int_0^1 f(x) dg(x) = \int_0^1 f(x) g'(x) \, dx\]
\[= \int_0^1 x^2 \cos x \, dx\]
\[= x^2 \sin x \bigg|_0^1 - \int_0^1 2x \sin x \, dx\]
\[= \sin 1 - \left[ -2x \cos x \right]_0^1 + \int_0^1 2 \cos x \, dx\]
\[= \sin 1 + 2 \cos 1 - 0 + 2 \sin x \bigg|_0^1\]
\[= 3 \sin 1 + 2 \cos 1.\]

14. \[\sum_{j=1}^{N} \left| \frac{\cos j^3}{j^3} \right| \leq \sum_{j=M}^{N} \frac{1}{j^{3/4}} \leq \varepsilon \] for \( M, N \) large enough by the integral test. So the series is uniformly Cauchy. Hence the series converges uniformly. Therefore the sum is a continuous function.