Solutions to HW 2

5.2.1 1. Let $a_j = n$ for $2^n \leq j \leq 2^{n+1}$.

Then there are strings of length $n$, $2j$, $8j$, etc., but the sequence diverges to $+\infty$.

2. If the $b_j$ are bounded then there is a subsequence $b_{j,k}$ that are constantly equal to some integer $c$. So now we are looking at

$$a_{j,k} = \frac{b_{j,k}}{j}.$$

If infinitely many of the $a_{j,k}$ are the same, then that constant subsequence converges to a rational number. Otherwise $a_{j,k}$ does not converge.

In any case, we see that if $\{b_j\}$ is bounded then the sequence cannot converge to an irrational number.

3. The rational numbers with denominator a power of 2 are dense in the real line. So every real number is the limit of such a sequence.
8. (2) If \( x_j \rightarrow x \), \( y_j \rightarrow y \) then \( x_j + y_j \rightarrow x + y \).

Proof: Let \( \varepsilon > 0 \). Choose \( N_1 \) so large that
\[ j \geq N_1 \Rightarrow |x_j - x| < \frac{\varepsilon}{2} \].
Choose \( N_2 \) so large that
\[ j \geq N_2 \Rightarrow |y_j - y| < \frac{\varepsilon}{2} \].

Let
\[ N = \max \{ N_1, N_2 \} \].

If
\[ j > N \]
then
\[ |(x_j + y_j) - (x + y)| \leq |x_j - x| + |y_j - y| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

So \( x_j + y_j \rightarrow x + y \).

(4) If \( x_j \rightarrow x \), \( y_j \rightarrow y \), \( b_j \neq 0 \), \( b \neq 0 \), then
\[ \frac{x_j}{b_j} \rightarrow \frac{x}{b} \].

Proof: Let \( \varepsilon > 0 \). Choose \( N_1 \) so large that
\[ j > N_1 \Rightarrow |x_j - x| < \frac{\varepsilon}{2|b|} \].
Choose \( N_2 \) so large that
\[ j > N_2 \Rightarrow |y_j - y| < \frac{\varepsilon}{2|b|} \].
Choose \( N_3 \) so large that
\[ j > N_3 \Rightarrow \frac{1}{b_j} \neq 0 \].

Let
\[ j > \max \{ N_1, N_2, N_3 \} \].

Then
\[ \left| \frac{x_j}{b_j} - \frac{x}{b} \right| = \left| \frac{\beta(x_j - x) - \alpha y_j}{\beta b_j} \right| \leq \frac{|\beta(x_j - x)|}{|\beta b_j|} + \frac{|\alpha y_j|}{|\beta b_j|} \leq \frac{\varepsilon}{2|b|} + \frac{|1|}{|b_j|} \cdot \frac{|b_j|}{|b|} = \frac{\varepsilon}{2|b|} + \frac{1}{|b|} \cdot \frac{|b_j|}{|b|} \leq \frac{\varepsilon}{2|b|} + \frac{1}{|b|^2} \cdot \frac{1}{|b|^2} \leq \frac{\varepsilon}{2|b|^2} + \frac{1}{|b|^2} \leq \frac{\varepsilon}{|b|^2} \].

So \( \frac{x_j}{b_j} \rightarrow \frac{x}{b} \).
10. Suppose not. Then \( \exists \varepsilon > 0 \) s.t. \( s \leq t - \varepsilon \) for all \( s \in S \). But then say \( s \leq t - \varepsilon \), which is a contradiction.

11. Suppose not. Then there exist \( \varepsilon > 0 \) and \( 2j_1 < 2j_2 < 2j_3 < \ldots \) such that \( |x_{2j_k} - x| > \varepsilon \forall k \). But then \( \{x_{2j_k}\} \) does not have a subsequence that converges to \( x \).

12. If the sequence is not Cauchy, then \( \exists \varepsilon > 0 \) s.t. \( |x_m - x_n| > \varepsilon \) for infinitely many large \( m, n \). Let \( N > \frac{1}{\varepsilon} \). If we choose \( N \) such pairs \( m_j, n_j \), then \( \sum |x_{m_j} - x_{n_j}| \geq N \cdot \varepsilon > \frac{1}{\varepsilon} \cdot \varepsilon = 1. \)  
   Contradiction.
Section 2.2

1. Let \(a_j\) be a decreasing sequence that is bounded below. So \(a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n\).

Thus \(a_1 \geq a_j \geq a_1\), \(\forall j\).

So \(\{a_j\}\) is a bounded sequence. By Bolzano-Weierstrass, \(\exists\) a subsequence \(a_{j_k}\) which converges to some finite \(\beta\).

Let \(\varepsilon > 0\). Choose \(N\) so large that \(k > N \Rightarrow |a_{j_k} - \beta| < \varepsilon\). But then, if \(j > j_k\), \(|a_j - \beta| < \varepsilon\). So the full sequence converges to \(\beta\).

2. Let \(\{a_j\}\) be an enumeration of the rationals. Consider the sequence

\(*\) \(9, 9, 9_2, 9, 9_2, 9_3, 9, 9_2, 9_3, 9_4, \ldots\)

If \(x\) is any real number then there is certainly a sequence of rationals with \(n_j \to x\). But the sequence \(\{a_j\}\) is a subsequence of \(*\). Same if \(x = \pm \infty\).
4. \( x_{j+1} = x_j - \frac{x_j^2 - 2}{2x_j} \)

Notice that:
\[
(x_{j+1})^2 = x_j^2 - (x_j^2 - 2) + (x_j^2 - 2) = 2 + (x_j^2 - 2)^2 \geq 2 \quad \forall j.
\]

So \( x_{j+1} = x_j - \frac{x_j^2 - 2}{2x_j} \leq x_j \). So sequence is decreasing.

(?) Shows that the sequence is bounded below.

So the sequence converges to some \( \alpha \).

We can write:
\[
x_{j+1} = \frac{x_j + \frac{1}{x_j}}{2}.
\]

Letting \( j \to \infty \) gives:
\[
\alpha = \frac{\alpha + \frac{1}{\alpha}}{2}.
\]
\[
\alpha^2 = \frac{\alpha^2 + 1}{2}.
\]
\[
\alpha^2 = \frac{\alpha^2 + 1}{2} + 1.
\]
\[
\alpha^2 = 1.
\]
\[
\alpha^2 = 2
\]
\[
\alpha = \sqrt{2} \text{ is the limit.}
\]

8. There are infinitely many elements \( n \mod \pi \)
all contained in the interval \([0, \pi]\). By
Bolzano–Weierstrass, there is a subsequence
\( n_j \mod \pi \) that converges to some \( \alpha \) mod \( \pi \).
But then \((n_j - n_0) \mod \pi \to 0 \mod \pi\).

So the elements \((n_j - n_0) \mod \pi\) are arbitrarily small. But then the elements 
\(k(n_j - n_0) \mod \pi\), \(k \in \mathbb{N}\) are dense in \([0, \pi]\).