5.2
1. Let $S$ be a set and $T = \{ t \in \mathbb{R} : |s - t| < \varepsilon \}$ for some $s \in S$.
   We claim that $T$ is open. Let $t \in T$.
   Then there is an $s \in S$ such that $|s - t| < \varepsilon$.
   Let $u$ be a point such that $|u - t| < \varepsilon - n$. Then
   
   $|u - s| \leq |u - t| + |t - s| < (\varepsilon - n) + \eta = \varepsilon$.
   So $u \in T$. Hence $(t - (\varepsilon - n), t + (\varepsilon - n)) \subseteq T$.
   So $T$ is open.

3. The set $[0, 1)$ is neither open nor closed.

5. Let $X_j = \{ j \to \infty \}$. Then $\bigcap_{j=1}^{\infty} X_j = \emptyset$.
   And each $X_j$ is closed.

7. Let $U_j = (-\frac{1}{j}, 1 + \frac{1}{j})$. Then each $U_j$ is open and
   $\bigcap_{j=1}^{\infty} U_j = [0, 1]$ which is closed.
§ 4.2

1. Let $S$ be any set of real numbers. The closure of $S$ is $\overline{S} = S \cup \partial S$. So obviously $S \subseteq \overline{S}$.

Let $x \in$ complement of $S$, so $x \notin S$ and $x \notin \partial S$. Thus there is a neighborhood of $x$ that does not intersect both $S$ and $\partial S$.

That neighborhood obviously cannot intersect $\partial S$. And it does not intersect $S$. So, it lies in $\mathcal{C}(S \cup \partial S)$. Thus $\mathcal{C}(S \cup \partial S)$ is open. Hence

$\overline{S} = S \cup \partial S$ is closed.

Let $y \in \overline{S}$. Then of course $y \in S$ and $y$ cannot be in the interior of $S$ because no neighborhood of $y$ lies entirely in $S$.

So $y \in \partial S$.

Now let $z \in \overline{S} \setminus S$. So no neighborhood of $z$ lies entirely in $S$. So every neighborhood of $z$ intersects $\partial S$. Since $\overline{S} = S \cup \partial S$, $z$ either lies in $S$ or in $\partial S$. In the first case, every neighborhood of $z$ contains a point of $S$ (namely $z$).

In the second case, every neighborhood of $z$ contains a point of $S$ (that is the definition of boundary point). So $z \in \partial S$. Thus $\overline{S} \subseteq S \cup \partial S$. 
2. \( S = \{ 1, \frac{1}{2}, \frac{1}{3}, \ldots \} \cup \{ 0 \} \).
\[ S = \emptyset \]
\[ \emptyset \subseteq S \]

3. Let \( E_j = [\frac{j}{j^2}, 1 - \frac{j}{j^2}] \). Then each \( E_j \) is closed and \( \bigcap_{j=1}^{\infty} E_j = (0, 1) \), which is open. Instead let \( F_j = [0, 1] \). Then each \( F_j \) is closed and \( \bigcap_{j=1}^{\infty} F_j = [0, 1] \), which is closed.

5. Let \( S \subseteq \mathbb{R} \), let \( s \in S \). So \( \exists \varepsilon > 0 \) such that \( (s - \varepsilon, s + \varepsilon) \subseteq S \). Let \( t \in (s - \varepsilon, s + \varepsilon) \). So \( |t - s| = \varepsilon < \varepsilon \). If \( |u - t| < \varepsilon - \varepsilon \), then
\[ |u - s| \leq |u - t| + |t - s| < (\varepsilon - \varepsilon) + \varepsilon = \varepsilon \] so \( u \in S \). Hence \( S \) is open.

Let \( S \) be open, let \( s \in S \). Then \( \exists \varepsilon > 0 \) so that \( (s - \varepsilon, s + \varepsilon) \subseteq S \). So \( s \in S \). Hence \( S \subseteq S \). Now let \( t \in S \). Then \( \exists \varepsilon > 0 \) such that \( (t - \varepsilon, t + \varepsilon) \subseteq S \). Hence \( t \in S \). Thus \( S \subseteq S \).

In conclusion, \( S = S \).

Conversely, assume \( S = \emptyset \). Let \( x \in S \). Then \( x \in S \). So \( \exists \varepsilon > 0 \) such that \( (x - \varepsilon, x + \varepsilon) \subseteq S \). So \( S \) is open.
Eq. 3

1. Let K be compact and E closed. Then K\(\cap\)E is closed and bounded, so K\(\cap\)E is closed. Also K\(\cap\)E \(\subseteq\) K, so K\(\cap\)E is bounded. Hence K\(\cap\)E is compact.

2. Let K be compact and \(U\) open such that \(U \supseteq K\). If \(k \in K\) then \(k \in U\) so \(\exists \varepsilon > 0\) such that \(k - \varepsilon, k + \varepsilon \subseteq U\). Thus let \(J_k = (k - \varepsilon/2, k + \varepsilon/2)\). Then the intervals \(J_k\) form an open cover of K. So there is a finite subcover

\[
(k_1 - \varepsilon_1/2, k_1 + \varepsilon_1/2), (k_2 - \varepsilon_2/2, k_2 + \varepsilon_2/2), \ldots, (k_m - \varepsilon_m/2, k_m + \varepsilon_m/2).
\]

Let \(\varepsilon = \min \{\varepsilon_1/2, \varepsilon_2/2, \ldots, \varepsilon_m/2\}\).

If \(x \in K\) is any point then \(x \in J_{k_l}\) for some \(l = 1, \ldots, m\). So if \(|t - x| < \varepsilon\) then

\[
|t - k_l| \leq |t - x| + |x - k_l| < \varepsilon + \varepsilon/2 = \varepsilon/2.
\]

Hence \(t \in U\).
4. Let \( K \) be a compact set. For each \( k \in K \), let \( I_k = (k - \delta, k + \delta) \). Then the \( I_k \) form an open cover of \( K \). So there is a finite subcover \( I_{k_1}, I_{k_2}, \ldots, I_{k_l} \) of \( K \). That is what we seek.

8. If \( K \) is compact then \( K \) is closed and bounded. So \( cK \) is open and unbounded. So \( cK \) is not compact.

\[ \text{4.4} \]

1. We remove one set of length \( \frac{1}{3} \)
2. two sets of length \( \frac{1}{3} \)
3. four sets of length \( \frac{1}{3} \)
4. eight sets of length \( \frac{1}{3} \)
   
   etc.

So, the total length of all intervals removed is

\[
\sum_{j=1}^{\infty} \frac{2^{-j-1}}{5^j} = \frac{1}{5} \sum_{j=1}^{\infty} \left( \frac{2}{5} \right)^{j-1} = \frac{1}{5} \cdot \frac{2}{\frac{2}{5}} = \frac{1}{5} \cdot \frac{10}{3} = \frac{2}{3}.
\]

Thus, the constructed Cantor-like set has length \( \frac{2}{3} \).
We can assign a set of 0s and 1s to each element of this Cantor-like set, just as we did in the past. So the new Cantor set is uncountable.

This set is definitely different from the Cantor ternary set. After all, it has a different length.

3. Let $U = (0, 1)$ and $V = (1, 2)$. These are disjoint open sets. But if $\varepsilon > 0$, then $1 - \frac{\varepsilon}{3} \in U$ and $1 + \frac{\varepsilon}{3} \in V$ and

\[
|1 - \frac{\varepsilon}{3} - (1 + \frac{\varepsilon}{3})| = \frac{2\varepsilon}{3} < \varepsilon.
\]

So $\text{dist}(U, V) < \varepsilon$ for every $\varepsilon > 0$.

In conclusion, $\text{dist}(U, V) = 0$.

4. Let $0 < x < 1$, set $\varepsilon = \frac{x}{1 + 2^N}$. Construct a Cantor-like set by removing one intervel of length $\varepsilon$ at stage 1, two intervals of length $\varepsilon^2$ at stage 2, four intervals of length $\varepsilon^3$ at stage 3, etc.
So the total length of intervals summed is

\[
\sum_{j=1}^{\infty} 2^{j-1} \varepsilon^j = \varepsilon \sum_{j=1}^{\infty} (2\varepsilon)^{j-1}
\]

\[
= \varepsilon \sum_{j=0}^{\infty} (2\varepsilon)^j = \frac{\varepsilon}{1-2\varepsilon} = \frac{1}{1-2\left(\frac{1}{1+2\varepsilon}\right)} = \frac{1}{\varepsilon} = \lambda.
\]

Thus the complement of the Cantor-like set we are constructing has length \(\lambda\).

6. The Cantor set has length 0. So if \(x, c \in C\) and \(\varepsilon > 0\) then \((c-\varepsilon, c+\varepsilon)\) will intersect the complement. Hence \((c-\varepsilon, c+\varepsilon) \notin C\).

Thus \(C = \emptyset\). By the same token, if \(x \in C\) then for every \(\varepsilon > 0\) \((x-\varepsilon, x+\varepsilon)\) intersects both \(C\) and \(\overline{C}\). So \(x \notin C\).

Since \(C\) is closed, it contains all its boundary points. So \(\partial C = C\).
4.5

1. \( \{F_j\} = \bigcup_{j=1}^{\infty} F_j \). The each \( F_j \) is closed and bounded, hence compact. Also \( F_1 \supseteq F_2 \supseteq \cdots \).

So \( \bigcap_{j=1}^{\infty} F_j \) is non-empty. Therefore \( \bigcup_{j=1}^{\infty} F_j \) cannot be all of \( \mathbb{R} \).

3. Let \( E \) and \( F \) be perfect. Then each of \( E \times F \) is closed and every point of each set is an accumulation point. It follows that \( E \times F \) is closed and each point is an accumulation point. In particular, if \( (e, f) \in E \times F \) then \( \exists e_j \in E \) s.t. \( e_j \to e \) and \( \exists f_j \in F \) s.t. \( f_j \to f \). So \( (e_j, f_j) \to (e, f) \).

6. A connected set is a interval. All intervals \( [a, b] \) are perfect. The other are not. So \( [a, b) \), \( (a, b] \) and \( (a, b) \) are connected and not perfect. The set \( (a, b) \) is imperfect because its complement \( (-\infty, a] \cup [b, \infty) \) is perfect.
It is very difficult to describe all imperfect sets.

3. The interior of a perfect set will still have every point as accumulation point, but it will not be closed.