# Math 4121
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Figure: This is your instructor.

# The Lebesgue Integral

# Measurable Sets

Throughout this book we use the notation $\equiv$ to mean "is defined to be."

A *$\sigma$-algebra* of sets in $\mathbb{R}$ is a collection $\mathcal{X}$ of sets satisfying these axioms:

**(a)** $\emptyset, \mathbb{R}$ both belong to $\mathcal{X}$;

**(b)** If $A \in \mathcal{X}$, then ${}^c A \equiv \mathbb{R} \setminus A$ belongs to $\mathcal{X}$;

**(c)** If $\{A_j\}$ is a sequence of sets in $\mathcal{X}$, then $\cup_j A_j$ belongs to $\mathcal{X}$.

We call the ordered pair $(\mathbb{R}, \mathcal{X})$ a *measure space* (later on we shall augment this definition). Any set that is an element of $\mathcal{X}$ is called a *measurable set*. We will sometimes consider a measure space with abstract $\sigma$-algebra $\mathcal{X}$ on an abstract set $X$ rather than the more specific Borel sets on $\mathbb{R}$ (see below for a discussion of the Borel sets).

We may use de Morgan's laws of logic, together with properties **(b)** and **(c)** of a $\sigma$-algebra, to see that the countable intersection of measurable sets is measurable. Namely,

$$\bigcap E_j = {}^c\left(\bigcup {}^c E_j\right) .$$

If each $E_j$ is measurable, then the set on the right is measurable because it is formed with complementation and union. Hence the set on the left is measurable.

**Example:** We now present several examples of $\sigma$-algebras.

**I.** Let $\mathcal{X} = \{\text{all subsets of } \mathbb{R}\}$. Then it is straightforward to verify properties **(a)**, **(b)**, **(c)** of a $\sigma$-algebra.

**II.** Recall that a set is denumerable if it is either empty or finite or countable. Let $\mathcal{X}$ be those subsets of $\mathbb{R}$ which are either denumerable or have denumerable complement. Then it is easy to check **(a)**, **(b)**, **(c)** of a $\sigma$-algebra.

**III.** For us this will be the most important example of a $\sigma$-algebra. Namely, let $\mathcal{B}$ be the $\sigma$-algebra generated by the collection of open intervals. That is to say, we are considering all sets that can be obtained by taking **(i)** finite or countable unions of open intervals, **(ii)** finite or countable intersections of open intervals, or **(iii)** finite or countable unions or intersections of sets of types **(i)** or **(ii)**. This $\sigma$-algebra is called the *Borel sets*.

**IV.** This last example is a slight extension of Example **III** that is useful for measure theory. Treat the points $-\infty$ and $+\infty$ as formal objects. If $E$ is a Borel set as in **III**, then set

$$E_1 = E \cup \{-\infty\}\,,$$

$$E_2 = E \cup \{+\infty\}\,,$$

$$E_3 = E \cup \{-\infty, +\infty\}\,.$$

Now let $\widehat{\mathcal{B}}$ be all Borel sets together with all sets $E_1$, $E_2$, $E_3$ that are obtained from all possible Borel sets $E$. It is straightforward to check that this new $\widehat{\mathcal{B}}$ is a $\sigma$-algebra. We call this $\sigma$-algebra the *extended Borel sets*, and we often denote it by $\widehat{\mathcal{B}}$.

It will turn out that the collection of sets to which we can unambiguously assign a length or measure will form a $\sigma$-algebra. That $\sigma$-algebra will be very closely related to **III** and **IV** in the last example.

**Definition:** Let $\mathcal{X}$ be a $\sigma$-algebra on $\mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be $\mathcal{X}$*-measurable* if, for each real number $\alpha$, the set

$$\{x \in \mathbb{R} : f(x) > \alpha\} \tag{1}$$

belongs to $\mathcal{X}$.

**Remark:** If $S$ is the nonmeasurable set constructed earlier, then the function $f(x) = \chi_S$ will *not* be measurable. We will not be able to integrate this function $f$.

**Lemma:** The following four statements are equivalent for a function
$f : \mathbb{R} \to \mathbb{R}$ and a $\sigma$-algebra $\mathcal{X}$ on a set $X$.

**(a)** For every $\alpha \in \mathbb{R}$, the set $X_\alpha \equiv \{x \in \mathbb{R} : f(x) > \alpha\}$ belongs to $\mathcal{X}$.

**(b)** For every $\alpha \in \mathbb{R}$, the set $Y_\alpha \equiv \{x \in \mathbb{R} : f(x) \leq \alpha\}$ belongs to $\mathcal{X}$.

**(c)** For every $\alpha \in \mathbb{R}$, the set $Z_\alpha \equiv \{x \in \mathbb{R} : f(x) \geq \alpha\}$ belongs to $\mathcal{X}$.

**(d)** For every $\alpha \in \mathbb{R}$, the set $W_\alpha \equiv \{x \in \mathbb{R} : f(x) < \alpha\}$ belongs to $\mathcal{X}$.

**Proof:** Since $X_\alpha$ and $Y_\alpha$ are complementary, statement **(a)** is equivalent to statement **(b)**. Likewise, statements **(c)** and **(d)** are equivalent.

If **(a)** holds, then $X_{\alpha-1/j}$ belongs to $\mathcal{X}$ for each positive integer $j$. Since

$$Z_\alpha = \bigcap_{j=1}^{\infty} X_{\alpha-1/j}\,, \tag{2}$$

it follows that $Z_\alpha \in \mathcal{X}$. Thus **(a)** implies **(c)**.

In the same fashion, since

$$X_\alpha = \bigcup_{j=1}^{\infty} Z_{\alpha+1/j}\,, \tag{3}$$

it follows that **(c)** implies **(a)**.

A similar argument shows that **(b)** and **(d)** are logically equivalent.

We conclude that all four statements are logically equivalent. □

It is useful to know that the collection of measurable functions is closed under standard arithmetic operations.

**Lemma:** Fix a $\sigma$-algebra $\mathcal{X}$. Let $f$ and $g$ be $\mathcal{X}$-measurable, real-valued functions and let $c$ be a real number. Then each of the functions

$$cf \ , \ f^2 \ , \ f + g \ , \ f \cdot g \ , \ |f|$$

is measurable.

**Proof:** For the first result, suppose without loss of generality that $c > 0$. Then, for $\alpha > 0$,

$$\{x \in \mathbb{R} : cf(x) > \alpha\} = \{x \in \mathbb{R} : f(x) > \alpha/c\} \in \mathcal{X}\,.$$

For the second result, also assume that $\alpha > 0$ (the case $\alpha \le 0$ is trivial). Then

$$\begin{aligned} &\{x \in \mathbb{R} : f^2(x) > \alpha\} \\ &\quad = \{x \in \mathbb{R} : f(x) > \sqrt{\alpha}\} \cup \{x \in \mathbb{R} : f(x) < -\sqrt{\alpha}\} \in \mathcal{X}\,. \end{aligned}$$

For the third result, and $\alpha > 0$, define the set

$$S_r = \{x \in \mathbb{R} : f(x) > r\} \cap \{x \in \mathbb{R} : g(x) > \alpha - r\}$$

for $r$ a rational number. Obviously $S_r \in \mathcal{X}$ for each $r$. By considering the cases **(i)** $f(x) > 0$ and $g(x) - \alpha < 0$, **(ii)** $f(x) < 0$ and $g(x) - \alpha > 0$, and **(iii)** $f(x) > 0$ and $g(x) - \alpha > 0$, one may check that

$$\{x \in \mathbb{R} : (f+g)(x) > \alpha\} = \bigcup_{r \text{ rational}} S_r \in \mathcal{X} .$$

Thus $\{x \in \mathbb{R} : (f+g)(x) > \alpha\}$ lies in $\mathcal{X}$, so $f + g$ is measurable.

For the fourth result, observe that

$$f \cdot g = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right] .$$

Thus the measurability of $f \cdot g$ follows from the first three results.

For the fifth assertion, assume as before that $\alpha > 0$. Then

$$\{x \in \mathbb{R} : |f(x)| > \alpha\} = \{x \in \mathbb{R} : f(x) > \alpha\} \cup \{x \in \mathbb{R} : f(x) < -\alpha\} \in \mathcal{X} . \qquad \square$$