Math 4121
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Figure: This is your instructor.

The Lebesgue Integral

What is a Measure?

Example: Let f be a function from $\mathbb{R}$ to $\mathbb{R}$. Define

$$f^+(x) = \max\{f(x), 0\} = \frac{f(x) + |f(x)|}{2}$$

and

$$f^-(x) = \max\{-f(x), 0\} = \frac{|f(x)| - f(x)}{2} .$$

We think of f^+ as the *positive part* of f and f^- as the *negative part* of f. Observe that $f = f^+ - f^-$.

In view of the preceding results, we see immediately that f is measurable if and only if both f^+ and f^- are measurable.

In dealing with sequences of measurable functions, it is often convenient to consider suprema, infima, limsup, liminf, etc. Therefore we want to allow functions to take values in the extended reals (i.e., to take the values $+\infty$ and $-\infty$ as well as the usual real values). We wish to discuss measurability for functions taking values in the extended reals. In what follows, we denote the extended reals by $\widehat{\mathbb{R}}$.

Definition: Let f be a function from the reals to the extended reals. Let $\widehat{\mathcal{B}}$ be the extended Borel sets. We say that f is $\widehat{\mathcal{B}}$-measurable if, for each real number α, the set

$$\{x \in \mathbb{R} : f(x) > \alpha\}$$

lies in $\widehat{\mathcal{B}}$.

Notice that, if f is a $\widehat{\mathcal{B}}$-measurable function that takes values in the extended reals $\widehat{\mathbb{R}}$, then

$$\{x \in \mathbb{R} : f(x) = +\infty\} = \bigcap_{j=1}^{\infty} \{x \in \mathbb{R} : f(x) > j\}$$

and

$$\{x \in \mathbb{R} : f(x) = -\infty\} = \mathbb{R} \setminus \left[\bigcup_{j=1}^{\infty} \{x \in \mathbb{R} : f(x) > -j\} \right] .$$

Thus both these sets belong to $\widehat{\mathcal{B}}$.

The following somewhat technical lemma is often useful in dealing with extended real-valued functions.

Lemma: Let f be an extended real-valued function. Then f is $\widehat{\mathcal{B}}$-measurable if and only if the sets

$$E = \{x \in \mathbb{R} : f(x) = +\infty\}$$

and

$$F = \{x \in \mathbb{R} : f(x) = -\infty\}$$

belong to $\widehat{\mathcal{B}}$ and the real-valued function f^* defined by

$$f^*(x) = \begin{cases} f(x) & \text{if} \quad x \notin E \cup F\,, \\ 0 & \text{if} \quad x \in E \cup F\,. \end{cases}$$

is $\mathcal{B}$-measurable.

Proof: If f is $\widehat{\mathcal{B}}$-measurable, then it has already been shown that E and F belong to $\widehat{\mathcal{B}}$. Let $\alpha \in \mathbb{R}$ and $\alpha \geq 0$. Then

$$\{x \in \mathbb{R} : f^*(x) > \alpha\} = \{x \in \mathbb{R} : f(x) > \alpha\} \setminus E\,.$$

If instead $\alpha < 0$, then

$$\{x \in \mathbb{R} : f^*(x) > \alpha\} = \{x \in \mathbb{R} : f(x) > \alpha\} \cup F\,.$$

Thus f^* is $\mathcal{B}$-measurable.

For the converse, assume that $E, F \in \widehat{\mathcal{B}}$ and also that f^* is $\mathcal{B}$-measurable.

Let $\alpha \geq 0$. Then

$$\{x \in \mathbb{R} : f(x) > \alpha\} = \{x \in \mathbb{R} : f^*(x) > \alpha\} \cup E .$$

Also, if $\alpha < 0$, then

$$\{x \in \mathbb{R} : f(x) > \alpha\} = \{x \in \mathbb{R} : f^*(x) > \alpha\} \setminus F .$$

Thus f is $\widehat{\mathcal{B}}$-measurable. □

We see that, if f is extended real-valued and measurable, then the functions

$$cf \ , \ f^2 \ , \ |f| \ , \ f^+ \ , \ f^-$$

are all measurable.

Remark: Just a few comments about arithmetic in the extended reals.

By convention, we declare that $0 \cdot (+\infty) = 0 \cdot (-\infty) = 0$. If extended real-valued, measurable functions f and g take the values $+\infty$ and $-\infty$ respectively at a point x or the values $-\infty$ and $+\infty$ respectively at a point x, then the quantity $f(x) + g(x)$ is *not* well defined. But we may declare the value to be zero at such a point, and the resulting function $f + g$ will then be well defined. If f and g *both* take the value $+\infty$ at the point x, then we understand that $(f + g)(x) = +\infty$. Likewise for $-\infty$. Finally, in these last circumstances, $f(x) \cdot g(x) = +\infty$.

We would next like to see that the collection of measurable functions is closed under various limiting operations.

We begin by recalling some definitions.

Definition: Let $\{x_j\}$ be a sequence of real numbers. For $k = 1, 2, \ldots,$ we set

$$y_k = \inf\{x_j : j \in \mathbb{N}, j \geq k\}$$

and

$$z_k = \sup\{x_j : j \in \mathbb{N} : j \geq k\}.$$

Set

$$y = \lim_{k\to\infty} y_k = \sup\{y_k : k \in \mathbb{N}\}$$

and

$$z = \lim_{k\to\infty} z_k = \inf\{z_k : k \in \mathbb{N}\}.$$

Of course y and z are elements of the extended reals. We call y the *limit inferior* or *liminf* of the sequence $\{x_j\}$ and denote it by $\liminf x_j$. We call z the *limit superior* or *limsup* of the sequence $\{x_j\}$ and denote it by $\limsup x_j$.

In fact $\liminf x_j$ is the least of all subsequential limits of $\{x_j\}$ and $\limsup x_j$ is the greatest of all subsequential limits of $\{x_j\}$.

Proposition: Let $\{f_j\}$ be a sequence of $\mathcal{X}$-measurable functions. Define

$$f(x) = \inf_j f_j(x) \qquad , \qquad g(x) = \sup_j f_j(x) \quad ,$$

$$\underline{f}(x) = \liminf_{j\to\infty} f_j(x) \qquad , \qquad \overline{g}(x) = \limsup_{j\to\infty} f_j(x) \,.$$

Then $f, g, \underline{f}$, and $\overline{g}$ are all $\mathcal{X}$-measurable.

Proof: Let $\alpha \in \mathbb{R}$. Notice that

$$\{x \in \mathbb{R} : f(x) \geq \alpha\} = \bigcap_{j=1}^{\infty} \{x \in \mathbb{R} : f_j(x) \geq \alpha\}$$

and

$$\{x \in \mathbb{R} : g(x) > \alpha\} = \bigcup_{j=1}^{\infty} \{x \in \mathbb{R} : f_j(x) > \alpha\} .$$

Thus f and g are measurable when the f_j are.

Next observe that

$$\underline{f}(x) = \sup_{j \geq 1} \left\{ \inf_{m \geq j} f_m(x) \right\}$$

and

$$\overline{g}(x) = \inf_{j \geq 1} \left\{ \sup_{m \geq j} f_m(x) \right\} .$$

This implies that $\underline{f}$ and $\overline{g}$ are measurable. □

Corollary: If $\{f_j\}$ is a sequence of $\mathcal{X}$-measurable functions on $\mathbb{R}$ which converges to a function f on $\mathbb{R}$, then f is $\mathcal{X}$-measurable.

Proof: We note that

$$f(x) = \lim_{j\to\infty} f_j(x) = \liminf_{j\to\infty} f_j(x) = \limsup_{j\to\infty} f_j(x)$$

and the result follows. □

Remark: Let us say a few words now about products of measurable functions. So assume that f and g are $\mathcal{X}$-measurable. For $j \in \mathbb{N}$, let f_j be the truncation of f defined as follows:

$$f_j(x) = \begin{cases} f(x) & \text{if} \quad |f(x)| \le j\,, \\ j & \text{if} \quad f(x) > j\,, \\ -j & \text{if} \quad f(x) < -j\,. \end{cases}$$

Define g_j similarly.

It is easy to check that f_j and g_j are measurable (recall that characteristic functions are measurable and products and sums preserve measurability). It follows from the above lemma that $f_k \cdot g_j$ is measurable. Since

$$f(x) \cdot g_j(x) = \lim_{k \to \infty} f_k(x) \cdot g_j(x) \quad \text{for } x \in \mathbb{R}\,,$$

it follows from the corollary above that $f \cdot g_j$ is measurable. Reasoning similarly,

$$(f \cdot g)(x) = f(x) \cdot g(x) = \lim_{j \to \infty} f(x) \cdot g_j(x) \quad \text{for } x \in \mathbb{R}\,.$$

Thus our corollary implies once again that $f \cdot g$ is measurable.

Recall that, if E is a set, then χ_E is the characteristic function of E: it takes the value 1 at points of E and it takes the value 0 otherwise.

Definition: Let $E_1, E_2, \ldots, E_k$ be pairwise disjoint measurable sets. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be real numbers. The function

$$s(x) = \sum_{j=1}^{k} \alpha_j \, \chi_{E_j}(x)$$

is called a *simple function*.

Proposition: Let f be a nonnegative $\mathcal{X}$-measurable function. Then there is a sequence $\{s_k\}$ of simple functions with these properties:

(a) $0 \leq s_k(x) \leq s_{k+1}(x)$ for $x \in \mathbb{R}$ and $k \in \mathbb{N}$;

(b) $f(x) = \lim_{k\to\infty} s_k(x)$ for each $x \in \mathbb{R}$;

Proof: Fix a natural number k. For $j = 0, 1, \ldots, k \cdot 2^k - 1$, we let

$$S_{j,k} = \{x \in \mathbb{R} : j2^{-k} \leq f(x) < (j+1)2^{-k}\} .$$

Also, if $j = k \cdot 2^k$, set $S_{j,k} = \{x \in \mathbb{R} : f(x) \geq k\}$.

Notice that, for fixed k, the sets S_{jk} are pairwise disjoint for $j = 0, 1, \ldots, k \cdot 2^k$. Also each of these sets belongs to $\mathcal{X}$; and the union of the sets is all of $\mathbb{R}$. Now set

$$s_k(x) = j \cdot 2^{-k} \quad \text{for} \quad x \in S_{j,k} \ , \ j = 0, 1, \ldots, k \cdot 2^k .$$

Then certainly each s_k is measurable. And properties **(a)** and **(b)** of the proposition are now immediate. □