What is a Measure?
Figure: This is your instructor.
The Lebesgue Integral
What is a Measure?

In what follows we let $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$. We let $\hat{\mathbb{R}}$ denote the extended reals, and we let $\hat{\mathbb{R}}^+ = \{x \in \hat{\mathbb{R}} : x \geq 0\}$. Thus $\hat{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{+\infty\}$.

**Definition:** Let $\mathcal{X}$ be a $\sigma$-algebra on $\mathbb{R}$. A *measure* $\mu$ is a function $\mu : \mathcal{X} \to \hat{\mathbb{R}}^+$ such that

(a) $\mu(\emptyset) = 0$;

(b) If $E_1, E_2, \ldots$ are pairwise disjoint sets in $\mathcal{X}$, then

$$\mu \left( \bigcup_{j=1}^{\infty} E_j \right) = \sum_{j=1}^{\infty} \mu(E_j).$$  \hspace{1cm} (1)
Notice that we may obtain the value $+\infty$ in equation (1) only if either (i) one of the $\mu(E_j)$ equals $+\infty$ or (ii) the sum of the $\mu(E_j)$ is $+\infty$. If a given measure $\mu$ never takes on the value $+\infty$, then we call that measure finite. If instead $\mathbb{R} = \bigcup_j E_j$ and each $\mu(E_j)$ is finite, then we say that $\mu$ is $\sigma$-finite.
**Example:** Let us consider some examples of measures.

(a) Let $\mathcal{B}$ be the Borel $\sigma$-algebra on $\mathbb{R}$. Set $\mu(E) = 0$ for all Borel sets $E$. Then this $\mu$ is a measure (although not a very interesting one).

(b) Let $\mathcal{B}$ be the Borel $\sigma$-algebra on $\mathbb{R}$. Define $\mu(\emptyset) = 0$ and $\mu(E) = +\infty$ for every other Borel set $E$. Then $\mu$ is a measure. It is *not* $\sigma$-finite.
(c) Let $\mathcal{B}$ be the Borel $\sigma$-algebra on $\mathbb{R}$. Fix a point $P \in \mathbb{R}$. For $E$ a Borel set define

$$
\mu_P(E) = \begin{cases} 
0 & \text{if } P \notin E, \\
1 & \text{if } P \in E.
\end{cases}
$$

Then $\mu_P$ is a finite measure. We sometimes call this the \textit{unit measure} or \textit{point mass} or \textit{Dirac mass} concentrated at $P$.

(d) One of the main points of this course is to construct a measure $\mu$ on the Borel $\sigma$-algebra $\mathcal{B}$ in $\mathbb{R}$ which assigns to each interval $[a, b]$ or $(a, b)$ or $[a, b)$ or $(a, b]$ the measure $b - a$. This will be the famous \textit{Lebesgue measure} constructed by H. Lebesgue in 1902. This is not a finite measure, but it is $\sigma$-finite because the length of each interval $l_j = [j, j + 1]$ will be 1 and $\bigcup_j l_j = \mathbb{R}$. We will be developing Lebesgue measure in the remainder of this course.
(e) Let $g$ be a strictly monotone increasing function from $\mathbb{R}$ to $\mathbb{R}$. We will see later that there exists a measure $\mu$ that assigns to each interval $[a, b]$ or $(a, b)$ or $[a, b]$ or $(a, b]$ the measure $g(b) - g(a)$. This is the Borel-Stieltjes measure $\mu_g$ induced by $g$.

(f) Let $A = \{a_j\}_{j=1}^{\infty}$ be a countable set of real numbers. For a set $E$ in the Borel $\sigma$-algebra $\mathcal{B}$, we define $\mu(E)$ to be the number of elements of $A$ that lies in $E$. Then $\mu$ is a measure.
We will be developing Lebesgue measure in the remainder of this course. Now we prove some basic results about measures.

**Lemma:** Let $\mathcal{X}$ be a $\sigma$-algebra on $\mathbb{R}$. Let $\mu$ be a measure on $\mathcal{X}$. If $E, F \in \mathcal{X}$ and $E \subseteq F$, then $\mu(E) \leq \mu(F)$. Also, if $\mu(E) < \infty$, then $\mu(F \setminus E) = \mu(F) - \mu(E)$.
Proof: Write $F = E \cup (F \setminus E)$ and note that $E \cap (F \setminus E) = \emptyset$. It follows that

$$
\mu(F) = \mu(E) + \mu(F \setminus E).
$$

(2)

Since $\mu(F \setminus E) \geq 0$, we conclude that $\mu(F) \geq \mu(E)$. If $\mu(E) < \infty$, then we can subtract it from both sides of equation (2) to obtain the second assertion. □
Lemma:

Let \( \mu \) be a measure defined on a \( \sigma \)-algebra \( \mathcal{X} \) on \( \mathbb{R} \).

(a) If \( \{ E_j \}_{j=1}^{\infty} \) is an increasing sequence of sets (i.e., \( E_1 \subseteq E_2 \subseteq \cdots \)) in \( \mathcal{X} \), then

\[
\mu \left( \bigcup_{j=1}^{\infty} E_j \right) = \lim_{n \to \infty} \mu(E_n). \tag{1}
\]

(b) If \( \{ F_j \}_{j=1}^{\infty} \) is a decreasing sequence of sets (i.e., \( F_1 \supseteq F_2 \supseteq \cdots \)) in \( \mathcal{X} \) and if \( \mu(F_1) < +\infty \), then

\[
\mu \left( \bigcap_{j=1}^{\infty} F_j \right) = \lim_{n \to \infty} \mu(F_n). \tag{2}
\]
**Proof:** For part (a), if \( \mu(E_j) = +\infty \) for some \( j \), then both sides of equation (1) are \( +\infty \). Thus we may assume that \( \mu(E_j) < +\infty \) for all \( j \).

Let \( A_1 = E_1 \) and set \( A_j = E_j \setminus E_{j-1} \) for \( j > 1 \). Then \( \{A_j\} \) is a sequence of pairwise disjoint sets in \( \mathbb{R} \) so that

\[
E_j = \bigcup_{m=1}^{j} A_m \quad \text{and} \quad \bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} A_j.
\]

Because \( \mu \) is countably additive, we see that

\[
\mu \left( \bigcup_{j=1}^{\infty} E_j \right) = \mu \left( \bigcup_{j=1}^{\infty} A_j \right) = \sum_{j=1}^{\infty} \mu(A_j) = \lim_{n \to \infty} \sum_{j=1}^{n} \mu(A_j). \tag{3}
\]

By the last lemma, \( \mu(A_j) = \mu(E_j) - \mu(E_{j-1}) \) for \( j > 1 \). Hence the finite series on the righthand side of equation (3) telescopes and

\[
\sum_{j=1}^{n} \mu(A_j) = \mu(E_n).
\]
For part (b), set \( E_j = F_1 \setminus F_j \), so that \( \{E_j\} \) is an increasing sequence of sets in \( \mathcal{F} \). We may apply part (a) and the earlier lemma to conclude that

\[
\mu \left( \bigcup_{j=1}^{\infty} E_j \right) = \lim_{n \to \infty} \mu(E_n) = \lim_{n \to \infty} \left[ \mu(F_1) - \mu(F_n) \right] = \mu(F_1) - \lim_{n \to \infty} \mu(F_n). \tag{4}
\]
Since
\[ \bigcup_{j=1}^{\infty} E_j = F_1 \setminus \bigcap_{j=1}^{\infty} F_j, \]
we may conclude that
\[ \mu \left( \bigcup_{j=1}^{\infty} E_j \right) = \mu(F_1) - \mu \left( \bigcap_{j=1}^{\infty} F_j \right). \]

Combining (4) and (5) gives (2). \(\Box\)
**Remark:** Part (b) of the lemma is false without the hypothesis that \( \mu(F_1) < +\infty \). For consider the example of \( \mu \) being Lebesgue measure and \( F_1 = [1, +\infty), \ F_2 = [2, +\infty), \ldots, \ F_j = [j, +\infty), \ldots \). Then notice that this is indeed a decreasing collection of sets, but \( \mu(F_1) = +\infty \). And the lefthand side of (2) is \( \mu(\emptyset) = 0 \) while the righthand side is \( \lim_{n \to \infty} \mu(F_n) = +\infty \).

**Definition:** A *measure space* is a triple \((\mathbb{R}, \mathcal{X}, \mu)\), where \( \mathbb{R} \) is the real numbers, \( \mathcal{X} \) is a \( \sigma \)-algebra, and \( \mu \) is a measure.
And now an important and central piece of terminology.

**Definition:** We shall say that a certain property \((P)\) holds \(\mu\)-almost everywhere if there is a subset \(N \subseteq \mathbb{R}\) with \(\mu(N) = 0\) and so that \((P)\) holds on \(\mathbb{R} \setminus N\). For instance, we say that two functions \(f\) and \(g\) are equal \(\mu\)-almost everywhere precisely when \(f(x) = g(x)\) when \(x \not\in N\) and \(N\) has measure 0. In these circumstances we shall often write

\[
f = g \quad \mu\text{-a.e.}
\]

or sometimes

\[
f = g \quad \text{a.e.}
\]

when the measure is understood from context.
In a similar manner, we shall say that a sequence of functions \( \{f_j\} \) on \( \mathbb{R} \) converges \( \mu \)-almost everywhere if there is a set \( N \subseteq \mathbb{R} \) with \( \mu(N) = 0 \) and so that \( \lim_{j \to \infty} f_j(x) \) exists for all \( x \in \mathbb{R} \setminus N \). Call the limit function \( f \). In these circumstances we often write

\[
f = \lim_{j \to \infty} f_j(x) \quad \mu\text{-a.e.}
\]

or sometimes

\[
f = \lim_{j \to \infty} f_j(x) \quad \text{a.e.}
\]

when the measure is understood from context.