Math 4121
February 3, 2021 Lecture

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January 26, 2021

krantzphoto.jpg

Figure: This is your instructor.

The Lebesgue Integral

What is a Measure?

In what follows we let $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$. We let $\widehat{\mathbb{R}}$ denote the extended reals, and we let $\widehat{\mathbb{R}}^+ = \{x \in \widehat{\mathbb{R}} : x \geq 0\}$. Thus $\widehat{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{+\infty\}$.

Definition: Let $\mathcal{X}$ be a σ-algebra on $\mathbb{R}$. A *measure* μ is a function $\mu : \mathcal{X} \to \widehat{\mathbb{R}}^+$ such that

(a) $\mu(\emptyset) = 0$;

(b) If $E_1, E_2, \ldots$ are pairwise disjoint sets in $\mathcal{X}$, then

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)\,. \tag{1}$$

Notice that we may obtain the value $+\infty$ in equation (1) only if either **(i)** one of the $\mu(E_j)$ equals $+\infty$ or **(ii)** the sum of the $\mu(E_j)$ is $+\infty$. If a given measure μ never takes on the value $+\infty$, then we call that measure *finite*. If instead $\mathbb{R} = \cup_j E_j$ and each $\mu(E_j)$ is finite, then we say that μ is σ-finite.

Example: Let us consider some examples of measures.

(a) Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}$. Set $\mu(E) = 0$ for all Borel sets E. Then this μ is a measure (although not a very interesting one).

(b) Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}$. Define $\mu(\emptyset) = 0$ and $\mu(E) = +\infty$ for every other Borel set E. Then μ is a measure. It is *not* σ-finite.

(c) Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}$. Fix a point $P \in \mathbb{R}$. For E a Borel set define

$$\mu_P(E) = \begin{cases} 0 & \text{if} \quad P \notin E\,, \\ 1 & \text{if} \quad P \in E\,. \end{cases}$$

Then μ_P is a finite measure. We sometimes call this the *unit measure* or *point mass* or *Dirac mass* concentrated at P.

(d) One of the main points of this course is to construct a measure μ on the Borel σ-algebra $\mathcal{B}$ in $\mathbb{R}$ which assigns to each interval $[a, b]$ or (a, b) or $[a, b)$ or $(a, b]$ the measure $b - a$. This will be the famous *Lebesgue measure* constructed by H. Lebesgue in 1902. This is not a finite measure, but it is σ-finite because the length of each interval $I_j = [j, j + 1]$ will be 1 and $\cup_j I_j = \mathbb{R}$. We will be developing Lebesgue measure in the remainder of this course.

(e) Let g be a strictly monotone increasing function from $\mathbb{R}$ to $\mathbb{R}$. We will see later that there exists a measure μ that assigns to each interval $[a, b]$ or (a, b) or $[a, b)$ or $(a, b]$ the measure $g(b) - g(a)$. This is the *Borel-Stieltjes measure* μ_g *induced by* g.

(f) Let $A = \{a_j\}_{j=1}^{\infty}$ be a countable set of real numbers. For a set E in the Borel σ-algebra $\mathcal{B}$, we define $\mu(E)$ to be the number of elements of A that lies in E. Then μ is a measure.

We will be developing Lebesgue measure in the remainder of this course. Now we prove some basic results about measures.

Lemma: *Let $\mathcal{X}$ be a σ-algebra on $\mathbb{R}$. Let μ be a measure on $\mathcal{X}$. If $E, F \in \mathcal{X}$ and $E \subseteq F$, then $\mu(E) \leq \mu(F)$. Also, if $\mu(E) < \infty$, then $\mu(F \setminus E) = \mu(F) - \mu(E)$.*

Proof: Write $F = E \cup (F \setminus E)$ and note that $E \cap (F \setminus E) = \emptyset$. It follows that

$$\mu(F) = \mu(E) + \mu(F \setminus E)\,. \tag{2}$$

Since $\mu(F \setminus E) \geq 0$, we conclude that $\mu(F) \geq \mu(E)$. If $\mu(E) < \infty$, then we can subtract it from both sides of equation (2) to obtain the second assertion. □

Lemma:

Let μ be a measure defined on a σ-algebra $\mathcal{X}$ on $\mathbb{R}$.

(a) *If $\{E_j\}_{j=1}^{\infty}$ is an increasing sequence of sets (i.e., $E_1 \subseteq E_2 \subseteq \cdots$) in $\mathcal{X}$, then*

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{n\to\infty} \mu(E_n)\,. \tag{1}$$

(b) *If $\{F_j\}_{j=1}^{\infty}$ is a decreasing sequence of sets (i.e., $F_1 \supseteq F_2 \supseteq \cdots$) in $\mathcal{X}$ and if $\mu(F_1) < +\infty$, then*

$$\mu\left(\bigcap_{j=1}^{\infty} F_j\right) = \lim_{n\to\infty} \mu(F_n)\,. \tag{2}$$

Proof: For part **(a)**, if $\mu(E_j) = +\infty$ for some j, then both sides of equation (1) are $+\infty$. Thus we may assume that $\mu(E_j) < +\infty$ for all j.

Let $A_1 = E_1$ and set $A_j = E_j \setminus E_{j-1}$ for $j > 1$. Then $\{A_j\}$ is a sequence of pairwise disjoint sets in $\mathbb{R}$ so that

$$E_j = \bigcup_{m=1}^{j} A_m \quad \text{and} \quad \bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} A_j \,.$$

Because μ is countably additive, we see that

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(A_j) \,. \qquad (3)$$

By the last lemma, $\mu(A_j) = \mu(E_j) - \mu(E_{j-1})$ for $j > 1$. Hence the finite series on the righthand side of equation (3) telescopes and

$$\sum_{j=1}^{n} \mu(A_j) = \mu(E_n) \,.$$

For part **(b)**, set $E_j = F_1 \setminus F_j$, so that $\{E_j\}$ is an increasing sequence of sets in . We may apply part **(a)** and the earlier lemma to conclude that

$$\begin{aligned}\mu\left(\bigcup_{j=1}^{\infty} E_j\right) &= \lim_{n\to\infty} \mu(E_n) \\ &= \lim_{n\to\infty} [\mu(F_1) - \mu(F_n)] \\ &= \mu(F_1) - \lim_{n\to\infty} \mu(F_n) \,. \end{aligned} \tag{4}$$

Since

$$\bigcup_{j=1}^{\infty} E_j = F_1 \setminus \bigcap_{j=1}^{\infty} F_j \,,$$

we may conclude that

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \mu(F_1) - \mu\left(\bigcap_{j=1}^{\infty} F_j\right) \,. \tag{5}$$

Combining (4) and (5) gives (2). □

Remark: Part **(b)** of the lemma is false without the hypothesis that $\mu(F_1) < +\infty$. For consider the example of μ being Lebesgue measure and $F_1 = [1, +\infty)$, $F_2 = [2, +\infty)$, $\ldots$, $F_j = [j, +\infty)$,$\ldots$. Then notice that this is indeed a decreasing collection of sets, but $\mu(F_1) = +\infty$. And the lefthand side of (2) is $\mu(\emptyset) = 0$ while the righthand side is $\lim_{n\to\infty} \mu(F_n) = +\infty$.

Definition: A *measure space* is a triple $(\mathbb{R}, \mathcal{X}, \mu)$, where $\mathbb{R}$ is the real numbers, $\mathcal{X}$ is a σ-algebra, and μ is a measure.

And now an important and central piece of terminology.

Definition: We shall say that a certain property (P) holds *μ-almost everywhere* if there is a subset $N \subseteq \mathbb{R}$ with $\mu(N) = 0$ and so that (P) holds on $\mathbb{R} \setminus N$. For instance, we say that two functions f and g are equal μ-almost everywhere precisely when $f(x) = g(x)$ when $x \notin N$ and N has measure 0. In these circumstances we shall often write

$$f = g \quad \mu\text{-a.e.}$$

or sometimes

$$f = g \quad \text{a.e.}$$

when the measure is understood from context.

In a similar manner, we shall say that a sequence of functions $\{f_j\}$ on $\mathbb{R}$ converges μ-almost everywhere if there is a set $N \subseteq \mathbb{R}$ with $\mu(N) = 0$ and so that $\lim_{j\to\infty} f_j(x)$ exists for all $x \in \mathbb{R} \setminus N$. Call the limit function f. In these circumstances we often write

$$f = \lim_{j\to\infty} f_j(x) \quad \mu\text{-a.e.}$$

or sometimes

$$f = \lim_{j\to\infty} f_j(x) \quad \text{a.e.}$$

when the measure is understood from context.